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When Varignon meets Napoleon

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Abstract. In the well-known Napoleon's theorem, equilateral triangles are built either all inwards or all outwards on the sides of a triangle. The centers of these triangles then form an equilateral triangle. Thébault's theorem makes an analogous statement for parallelograms: If one erects squares on the sides (all inwards or all outwards), their centers form a square. Varignon's theorem applies to general quadrilaterals: The centers of the sides form a parallelogram. We show the following for general quadrilaterals: Erect directly similar triangles on the sides, *alternately* inwards and outwards. Then the vertices, the centers of gravity, the orthocenters, the centers of the inscribed circle, or any other centers of these triangles always form a parallelogram. We also explore some of the nice geometric properties of these parallelograms.

Mathematics Subject Classification. 51M04, 51M15.

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1. Introduction

There is a remarkable number of theorems that use non-regular polygons to establish surprising regular structures. One of the best known is Varignon's theorem [3, § 3.1].

Theorem 1 (Varignon). *The midpoints of the sides of an arbitrary quadrilateral form a parallelogram (see Fig. 1).*

The assertion follows directly from the midline theorem applied to the triangles consisting of a diagonal and two adjacent quadrilateral sides. In particular, the sides of the parallelogram are parallel to the diagonals of the quadrilateral and have half their lengths.

A more spectacular example is Napolon's theorem (see [3, § 3.3], and [7] for the story of how this theorem got its name).

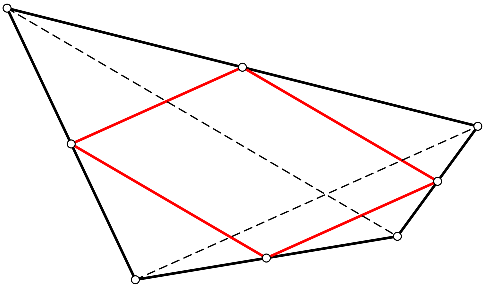

Figure 1 Varignon's Theorem

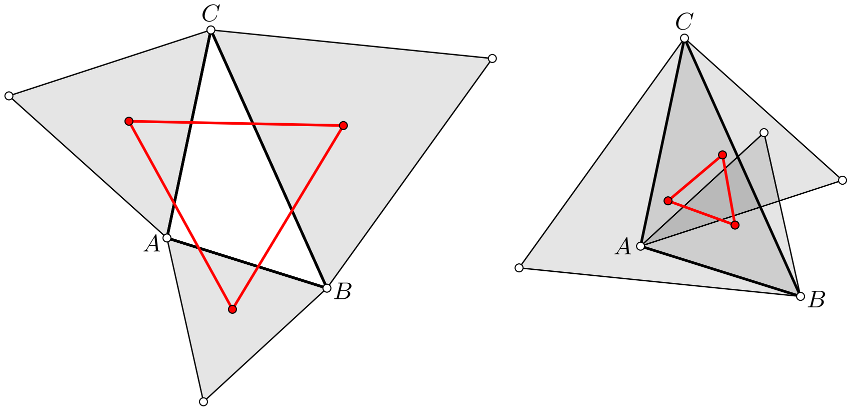

Figure 2 Napoleon's theorem. Outward version on the left, inward version on the right

Theorem 2 (Napoleon). *On each side of an arbitrary triangle ABC erect an equilateral triangle, either all inward or all outward. Then the centers of the three equilateral triangles build an equilateral triangle (see Fig. 2).*

Slightly less known is the theorem of Napoleon–Barlotti [1] which holds for affinely regular polygons, i.e., polygons that are the image of a regular polygon under an affine map.

Theorem 3 (Napoleon–Barlotti). *The centers of regular n-gons erected either all outward or all inward on the sides of an n-gon P form a regular n-gon if and only if P is affinely regular (see Fig. 3).*

The only affinely regular quadrilaterals are parallelograms. This special case of Theorem 3 is also known as Thébault's theorem.

We refer at this point to numerous other versions and generalisations of these results, e.g., the Petr–Douglas–Neumann theorem [6], the Jha–Savaran Generalization [5], the Dao Than Oai's generalization [9], Van Aubel's theorem [11], and Berendonk's theorem [2].

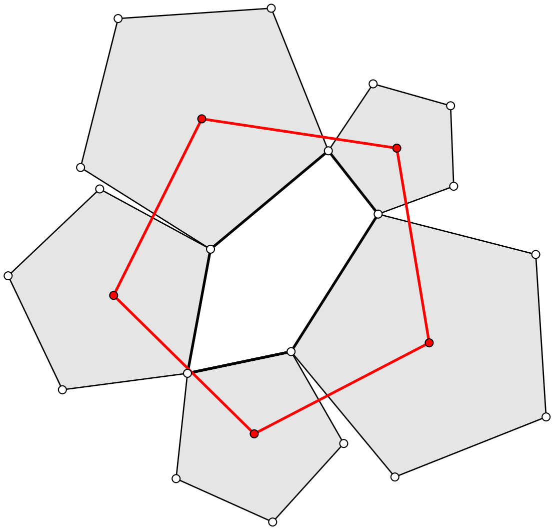

Figure 3 The theorem of Napoleon–Barlotti. The bold pentagon is affinely regular. This can be seen from the fact that the diagonals are parallel to the sides

2. A Varignon–Napoleon theorem for quadrilaterals

Although both Pierre Varignon and Napoleon Bonaparte were Frenchmen, they never met because they were not contemporaries. Nevertheless, they come together in the following theorem which first appeared in [4].[1] The Varignon theorem results as a special case of it, while the construction is inspired by Napoleon's theorem.

Theorem 4 (de Villiers). *Let $A_1A_2A_3A_4$ be an arbitrary quadrilateral. Erect directly similar triangles*[2] *$A_iA_{i+1}C_i$ on the sides of the quadrilateral, alternating on the outside and on the inside. Then $C_1C_2C_3C_4$ is a parallelogram (see Fig. 4).*

Notice that the theorem also holds for non-convex and self-intersecting quadrilaterals (see Fig. 5). The quadrilateral can even be degenerate, i.e., three or all four vertices can be collinear, as long as the four vertices are pairwise different. To clarify what inwards and outwards means in the general case, observe that the points A_{i-1}, A_i, A_{i+1} define a spiral similarity S_i with fixed point A_i such that $S_i(A_{i-1}) = A_{i+1}$. Then, the triangle $\triangle A_iA_{i+1}C_i$ is the image of the triangle $\triangle A_iA_{i-1}C_{i-1}$ under S_i. This interpretation works for all sorts of quadrilaterals. For the algebraic proof of Theorem 4 we will need a second spiral similarity T_i also with fixed point A_i and such that $T_i(A_{i-1}) = C_{i-1}$.

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[1] A dynamic version can be viewed on the webpage http://dynamicmathematicslearning.com/balance.html.

[2] Directly similar triangles are similar triangles that have the same orientation.

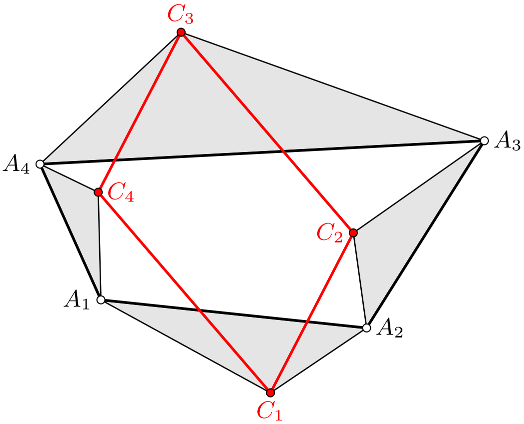

Figure 4 Theorem 4. The grey triangles are directly similar and are erected alternately inwards and outwards to the quadrilateral $A_1A_2A_3A_4$. Then $C_1C_2C_3C_4$ is a parallelogram

In what follows we will interpret points in the Euclidean plane as complex numbers in the complex plane $\mathbb{C}$. We use capital letters for the points and the corresponding lower case letter for the corresponding complex number. We also remark that indices are always meant cyclically. Then the spiral similarities are given by the following formulas:

$$S_i : \mathbb{C} \to \mathbb{C}, \;\; z \mapsto \frac{a_{i+1} - a_i}{a_{i-1} - a_i}(z - a_i) + a_i$$

$$T_i : \mathbb{C} \to \mathbb{C}, \;\; z \mapsto \frac{c_{i-1} - a_i}{a_{i-1} - a_i}(z - a_i) + a_i$$

By construction, we have $S_i(c_{i-1}) = c_i$, i.e.,

$$\frac{a_{i+1} - a_i}{a_{i-1} - a_i}(c_{i-1} - a_i) + a_i = c_i. \tag{1}$$

From this it follows immediately that

$$T_i(a_{i+1}) = \frac{c_{i-1} - a_i}{a_{i-1} - a_i}(a_{i+1} - a_i) + a_i = c_i. \tag{2}$$

Armed with these spiral similarities the proof of Theorem 4 is surprisingly simple.

Algebraic proof of Theorem 4. Fix c_1, then we get by (1)

$$\begin{aligned} c_2 = S_2(c_1) &= \frac{a_2a_3 - a_1a_2 + a_2c_1 - a_3c_1}{a_2 - a_1}, \\ c_3 = S_3(c_2) &= \frac{a_2a_3 - a_1a_4 - a_3c_1 + a_4c_1}{a_2 - a_1}, \\ c_4 = S_4(c_3) &= \frac{a_1a_2 - a_1a_4 - a_1c_1 + a_4c_1}{a_2 - a_1}. \end{aligned}$$

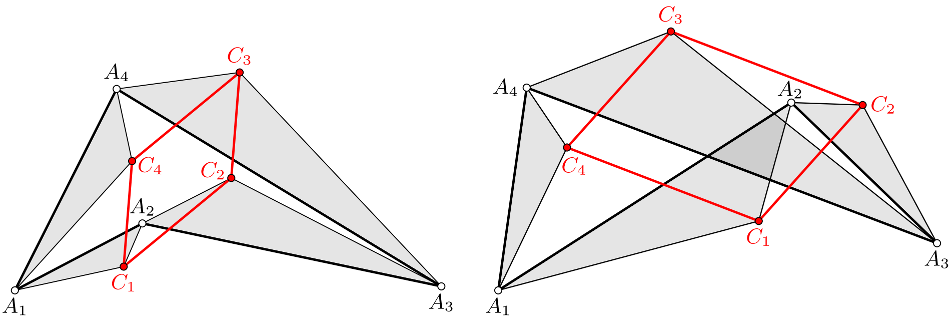

FIGURE 5 Theorem 4 for a non-convex (left) and a self-intersecting quadrilateral (right)

This immediately leads to

$$c_1 - c_4 = \frac{(a_4 - a_2)(a_1 - c_1)}{a_2 - a_1} = c_2 - c_3, \tag{3}$$

which proves the claim. □

If C_1 is chosen as midpoint of side A_1A_2 then Theorem 4 reduces to Varignon's theorem. An easy but at first sight surprising corollary is the following.

Corollary 1. *For $i = 1, \dots, 4$ let P_i be points with the same trilinear coordinates in the triangles $\triangle A_iA_{i+1}C_i$. Then $P_1P_2P_3P_4$ is a parallelogram.*

In particular, one can choose P_i as center of gravity, othocenter, center of the incircle or of the circumcircle, as foot of an altitude, or any triangle center. The corollary follows immediately from Theorem 4 by replacing the triangle $\triangle A_1A_2C_1$ by $\triangle A_1A_2P_1$.

In order to better understand the geometry of the parallelogram $C_1C_2C_3C_4$, we now take a closer look. Using the expressions we obtained in the proof of Theorem 4, we find

$$c_2 - c_1 = \frac{(a_3 - a_1)(a_2 - c_1)}{a_2 - a_1} = c_3 - c_4. \tag{4}$$

By (3) and (4) we get

$$\frac{c_4 - c_1}{c_2 - c_1} = \frac{(a_2 - a_4)(a_1 - c_1)}{(a_3 - a_1)(a_2 - c_1)} \tag{5}$$

and

$$\frac{c_1 - c_2}{c_3 - c_2} = \frac{(a_3 - a_1)(a_2 - c_1)}{(a_4 - a_2)(a_1 - c_1)}. \tag{6}$$

Using complex arguments this yields

$$\arg\frac{c_4 - c_1}{c_2 - c_1} = \arg\frac{a_2 - a_4}{a_3 - a_1} + \arg\frac{a_1 - c_1}{a_2 - c_1} \quad \text{mod } 2\pi \text{ and}$$
$$\arg\frac{c_1 - c_2}{c_3 - c_2} = \arg\frac{a_3 - a_1}{a_4 - a_2} + \arg\frac{a_2 - c_1}{a_1 - c_1} \quad \text{mod } 2\pi.$$

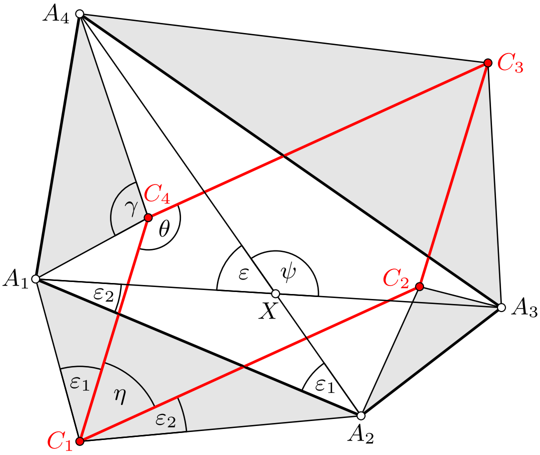

Figure 6 Proof of Theorem 5

If we denote the intersection of the diagonals by X, we get

$$\begin{aligned} \sphericalangle C_2C_1C_4 &= -\sphericalangle A_2XA_3 + \sphericalangle A_2C_1A_1 \quad \text{mod } 2\pi \text{ and} \\ \sphericalangle C_3C_2C_1 &= -\sphericalangle A_3XA_4 - \sphericalangle A_2C_1A_1 \quad \text{mod } 2\pi. \end{aligned} \tag{7}$$

Hence, the angles of the parallelogram are related to the angle of the triangle $\triangle A_1A_2C_1$ in point C_1 and the angles between the diagonals of the quadrilateral. The general situation is always fully described by (5) and (6), but the interpretation of the argument angles depends on the particular position of the points. We only work out the case of a convex quadrilateral.

Theorem 5. *Suppose that the quadrilateral $A_1A_2A_3A_4$ is convex, and assume that the triangle $A_1A_2C_1$ lies outside of the quadrilateral. Then, if the parallelogram $C_1C_2C_3C_4$ has the same orientation as the quadrilateral $A_1A_2A_3A_4$, its angles are given by*

$$\eta = \gamma - \varepsilon, \qquad \theta = 2\pi - (\gamma + \psi)$$

(see Fig. 6), in accordance with (7).

It is instructive to derive this result purely geometrically.

Geometric proof of Theorem 5. Using (2) we have $T_1(\triangle A_1A_2A_4) = \triangle A_1C_1C_4$ and hence $\sphericalangle A_4A_2A_1 = \sphericalangle C_4C_1A_1$. In Fig. 6 these angles are denoted by ε_1. Similarly, we have $T_2(\triangle A_2A_3A_1) = \triangle A_2C_2C_1$ and therefore $\sphericalangle A_2A_1A_3 = \sphericalangle A_2C_1C_2$. In the figure these angles are denoted by ε_2. Now we can read off $\eta = \gamma - (\varepsilon_1 + \varepsilon_2) = \gamma - \varepsilon$, and hence $\theta = \pi - \eta = 2\pi - (\gamma + \psi)$. □

It turns out that the parallelograms $C_1C_2C_3C_4$ constructed this way have many more nice geometric properties, which we want to explore in the next section.

3. Miquel comes along

We first want to show a result on quadrilaterals that is connected to another Frenchman, Auguste Miquel. Consider a quadrilateral $A_1A_2A_3A_4$ and construct an inscribed angle γ circle over each side, such that the major arc lies alternately outside and inside. These are exactly the circumcircles K_i of the triangles $\triangle A_iA_{i+1}C_i$ with angle γ in C_i which we considered above. We will call the circles K_i *Miquel* γ *circles* of the quadrilateral. Then we have the following.

Proposition 1. *The Miquel γ circles K_{i-1} and K_i meet in A_i and in a second point A'_i (see Fig. 7). Then the points A'_1 and A'_3 lie on the diagonal A_2A_4, and A'_2 and A'_4 lie on the diagonal A_1A_3. The quadrilateral $A'_1A'_2A'_3A'_4$ is similar to the quadrilateral $A_1A_2A_3A_4$, unless it degenerates to the intersection X of the diagonals.*

Proof. We only work out the proof for a convex quadrilateral $A_1A_2A_3A_4$. Other cases are analogous. Observe that the second intersection (different from A_1) of K_1 and the diagonal A_1A_3 must agree with A'_2 since the angles $\sphericalangle A_1A'_2A_2$ and $\sphericalangle A_2A'_2A_3$ are supplementary angles. A similar argument applies to the points A'_3, A'_4, and A'_1.

To see that the quadrilaterals $A_1A_2A_3A_4$ and $A'_1A'_2A'_3A'_4$ are similar, proceed as follows: The angles $\sphericalangle A'_1A'_2A_1$ and $\sphericalangle A'_1A_2A_1$ agree, since they are inscribed angles in K_1 over the points A_1 and A'_1. These angles are denoted by δ_1 in Fig. 7. The angles $\sphericalangle A_3A_2A'_3$ and $\sphericalangle A'_3A'_2A_3$ are supplementary angles, since there are opposite angles in the cyclic quadrilateral $A_3A_2A'_3A'_2$. Hence the angles $\sphericalangle A_1A'_2A'_3$ and $\sphericalangle A_3A_2A'_3$ agree. These angles are denoted by δ_2 in the figure. Similar arguments apply to the angles in A'_1, A'_4 and A'_3. Hence the quadrilaterals $A'_1A'_2A'_3A'_4$ and $A_1A_2A_3A_4$ have the same angles and the same angle between their diagonals and are therefore similar (see, e.g., [8,10]). ☐

Figure 7 indicates that the quadrilateral $A'_1A'_2A'_3A'_4$ has the opposite orientation compared to the quadrilateral $A_1A_2A_3A_4$. This remains the case if γ varies, and also if the quadrilateral $A_1A_2A_3A_4$ is not convex or self-intersecting.

Here is the connection to Miquel: The above proposition contains a special case of Miquel's six circle theorem, namely applied to the circles K_1, K_2, K_3, K_4 and the two diagonals.

The four points $A'_1, \ldots, A'_4$ will be important below, and will be called *pivot points*.

Theorem 6. *Let $A_1A_2A_3A_4$ be a quadrilateral, and $K_1, \ldots, K_4$ be Miquel γ circles over its sides. Consider a point C_1 on K_1 and construct directly similar triangles $A_iA_{i+1}C_i$ on the sides of the quadrilateral, alternating outwardly and inwardly. Then, independent of the point C_1 on $K_1 \setminus \{A_1, A_2\}$, the resulting parallelograms $C_1C_2C_3C_4$ have all the same angles, and the (extended) side C_iC_{i+1} passes through the pivot point A'_{i+1} (see Fig. 8).*

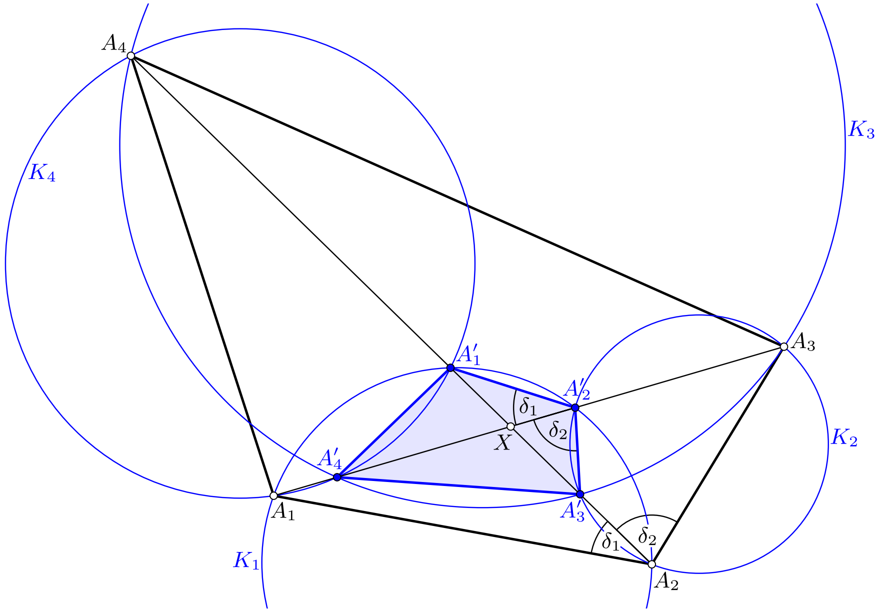

FIGURE 7 Proposition 1. The circles K_i are inscribed angle γ circles over each side of the quadrilateral $A_1A_2A_3A_4$, with major arc alternately outside and inside. Then the intersections A_i' of neighbouring circles lie on the diagonals, and the quadrilaterals $A_1'A_2'A_3'A_4'$ and $A_1A_2A_3A_4$ are similar

Note that for $C_1 = A_1$ or $C_1 = A_2$ the parallelogram degenerates to the line segment A_1A_3 or A_2A_4. Observe also that the parallelogram angle in C_1 changes to its supplement when C_1 moves from the major arc of K_1 to the minor arc.

Proof. Let C_1' be another point on $K_1 \setminus \{A_1, A_2\}$, i.e., the cross-ratio of a_1, a_2, c_1, c_1'

$$r := \frac{a_2 - c_1}{a_1 - c_1} : \frac{a_2 - c_1'}{a_1 - c_1'}$$

is a real number. For C_1' we get the parallelogram $C_1'C_2'C_3'C_4'$ for which also the relation (5) holds, i.e.,

$$\frac{c_4' - c_1'}{c_2' - c_1'} = \frac{(a_2 - a_4)(a_1 - c_1')}{(a_3 - a_1)(a_2 - c_1')}. \tag{8}$$

Dividing (8) by (5) we get

$$\frac{c_4' - c_1'}{c_2' - c_1'} : \frac{c_4 - c_1}{c_2 - c_1} = r \in \mathbb{R}.$$

If follows that the angles of the parallelogram in C_1 and C_1' either agree or are supplementary to each other.

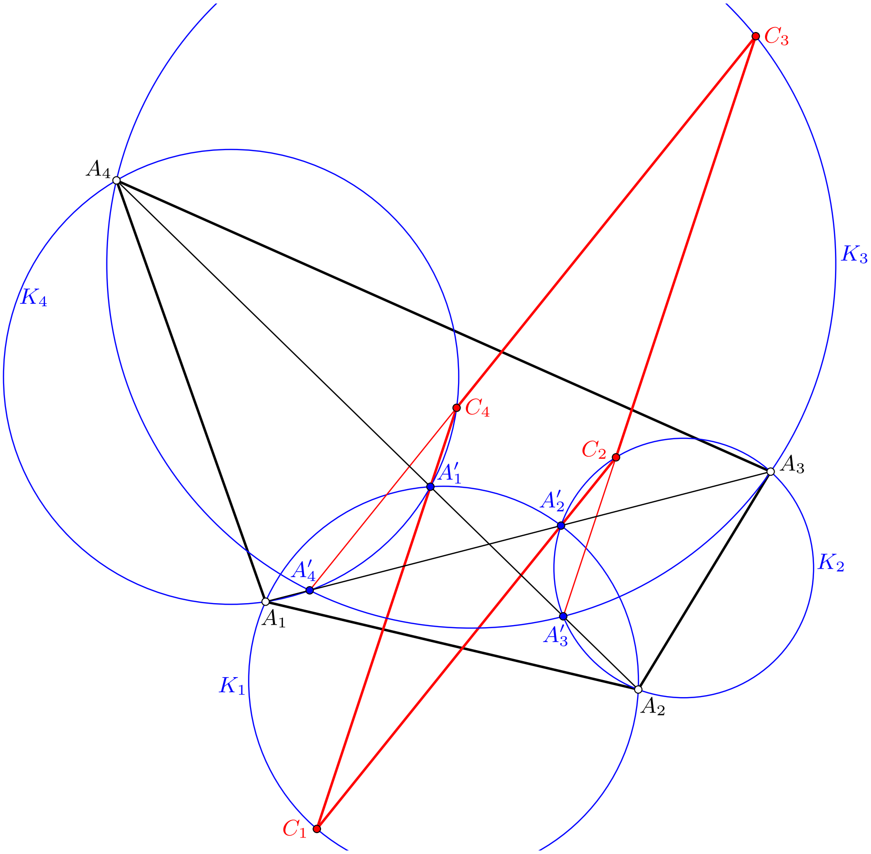

FIGURE 8 Theorem 6. The vertices C_2, C_3, C_4 of the parallelograms with vertex C_1 on K_1 lie on K_2, K_3, and K_4. The angles of the parallelogram are independent of the position of C_1 on K_1. And the sides of the parallelogram pass through the pivot points A_i'

To see that the points C_3, C_2, and A_3' are collinear, observe that

$$\sphericalangle A_2'C_2A_3' = \sphericalangle A_4'A_3A_3' = \sphericalangle A_4'C_3A_3'.$$

The first equality comes from the fact that both are inscribed angles in K_2 over the points A_2', A_3', and for the second equality observe that both are inscribed angles in K_3 over the points A_3', A_4'. Similar arguments show, that the points C_4, C_3, A_4', C_1, C_4, A_1', and C_2, C_1, A_2' are collinear. $\square$

There is even more to discover if we concentrate now on the area of the parallelogram. To this end we consider the Miquel circles $\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4$ of the quadrilateral $A_1A_2A_3A_4$ that pass through the intersection X of its diagonals. Let $\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4$ denote their centers (see Fig. 9).

Theorem 7. *Let $A_1A_2A_3A_4$ be a quadrilateral, with directly similar triangles $A_iA_{i+1}C_i$ on the sides of the quadrilateral, erected alternating outwardly and inwardly. If C_1 runs along a circle with center $\bar{M}_1$, then the points C_2, C_3, C_4 run along circles with centers $\bar{M}_2, \bar{M}_3, \bar{M}_4$ and the parallelograms $C_1C_2C_3C_4$ all share the same area (see Fig. 9). In particular, if C_1 runs along $\bar{K}_1$, then the parallelogram degenerates to a line segment.*

Proof. The area of the parallelogram $C_1C_2C_3C_4$ is given by

$$\mathcal{A}(c_1) = \frac{i}{2}\det\begin{pmatrix} 1 & 1 & 1 \\ c_4 & c_1 & c_2 \\ \bar{c}_4 & \bar{c}_1 & \bar{c}_1 \end{pmatrix} = \frac{i}{2}\frac{sc_1\bar{c}_1 + \bar{p}c_1 - p\bar{c}_1 + q}{(a_1 - a_2)(\bar{a}_1 - \bar{a}_2)}$$

where

$$\begin{aligned} p &= a_1(a_4 - a_2)(\bar{a}_3 - \bar{a}_1) + a_2(a_3 - a_1)(\bar{a}_2 - \bar{a}_4), \\ q &= \bar{a}_1\bar{a}_2(a_2a_3 - a_1a_4) + a_1\bar{a}_2\bar{a}_3(a_4 - a_2) + a_2\bar{a}_1\bar{a}_4(a_1 - a_3), \\ s &= (a_1 - a_3)(\bar{a}_4 - \bar{a}_2) + (a_2 - a_4)(\bar{a}_1 - \bar{a}_3). \end{aligned}$$

Here we used again the formulas for c_2 and c_4 from the proof of Theorem 4. Now we look for points C_1 such that $\mathcal{A}(c_1) = \text{const}$. This is the equation of a circle

$$(z - m)(\bar{z} - \bar{m}) = r^2 \tag{9}$$

with center

$$m = \frac{p}{s} = \frac{a_1(a_4 - a_2)(\bar{a}_3 - \bar{a}_1) + a_2(a_3 - a_1)(\bar{a}_2 - \bar{a}_4)}{(a_1 - a_3)(\bar{a}_4 - \bar{a}_2) + (a_2 - a_4)(\bar{a}_1 - \bar{a}_3)}. \tag{10}$$

This point M can be identified as the center of the circumcircle of the triangle A_1A_2X. Indeed, we have

$$x = \frac{(a_4 - a_2)(a_3\bar{a}_1 - a_1\bar{a}_3) + (a_3 - a_1)(a_2\bar{a}_4 - a_4\bar{a}_2)}{(a_2 - a_4)(\bar{a}_3 - \bar{a}_1) - (a_1 - a_3)(\bar{a}_4 - \bar{a}_2)}.$$

The equation of the circle through the points A_1, A_2, X is given by

$$\det\begin{pmatrix} z\bar{z} & z & \bar{z} & 1 \\ a_1\bar{a}_1 & a_1 & \bar{a}_1 & 1 \\ a_2\bar{a}_2 & a_2 & \bar{a}_2 & 1 \\ x\bar{x} & x & \bar{x} & 1 \end{pmatrix} = 0.$$

To see this, observe that this equation is clearly satisfied by $z = a_1, z = a_2$, and $z = x$, and by expanding the determinant and dividing by the coefficient of $z\bar{z}$ the equation takes the form (9). The coefficient of $z\bar{z}$ in this determinant is the minor

$$\det\begin{pmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ x & \bar{x} & 1 \end{pmatrix}$$

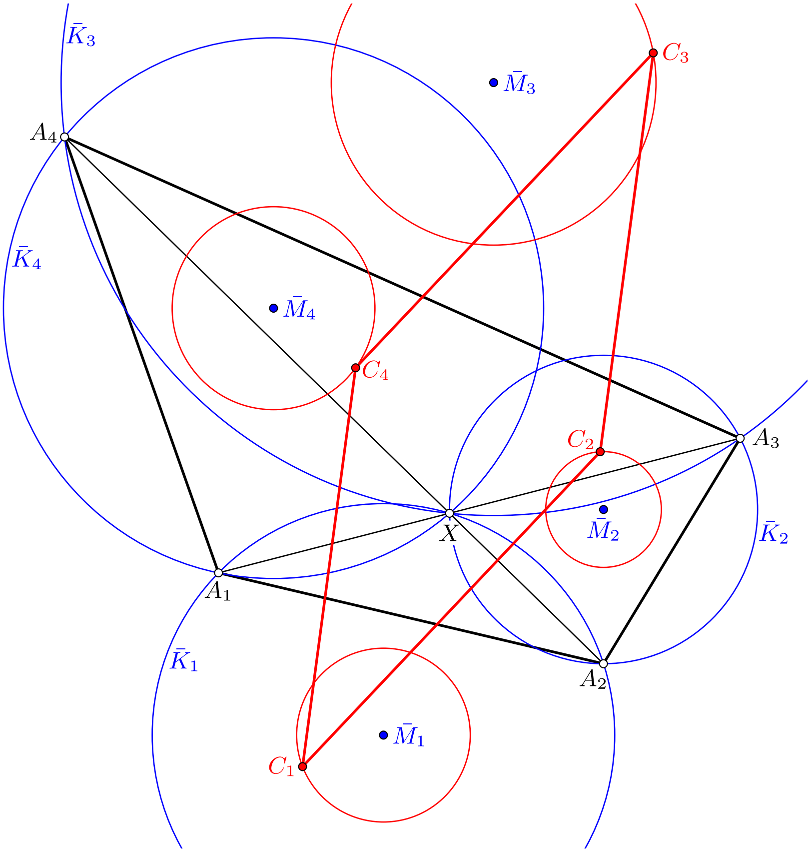

FIGURE 9 Theorem 7. If C_1 moves along a circle with center $\bar{M}_1$, then the points C_2, C_3, C_4 move along circles with centers $\bar{M}_2, \bar{M}_3, \bar{M}_4$, and all parallelograms $C_1C_2C_3C_4$ have the same area

and the coefficient of $\bar{z}$ is the minor

$$\det \begin{pmatrix} a_1\bar{a}_1 & a_1 & 1 \\ a_2\bar{a}_2 & a_2 & 1 \\ x\bar{x} & x & 1 \end{pmatrix}.$$

The center of the circle through A_1, A_2, X is then the negative quotient of these two determinants, and one verifies that its value indeed agrees with the expression in (10).

If $C_1 = A_1$, then $C_1 = C_4$ and $C_2 = C_3$ and the parallelogram degenerates to a line segment. Consequently, this also happens for any C_1 on $\bar{K}_1$. □

A nice special case is the following observation, which follows if we shrink the red circles in Fig. 9 to their centers. This brings us back to our starting point, the Varignon parallelogram.

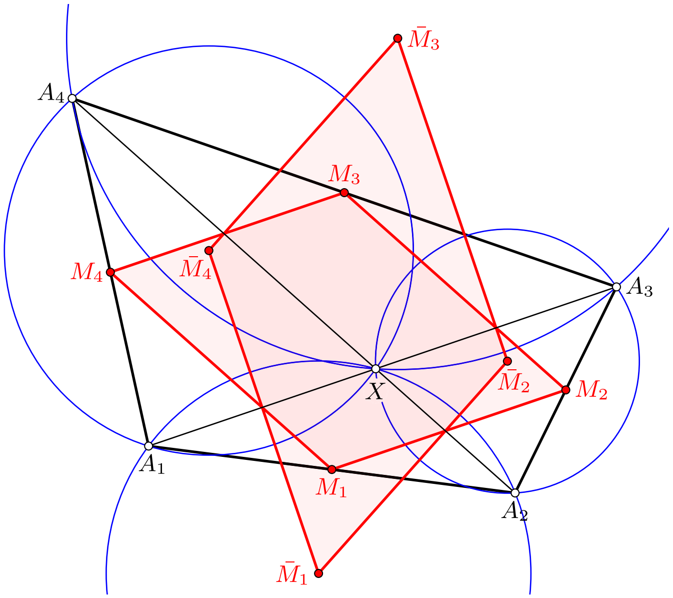

FIGURE 10 Corollary 2: The Varignon parallelogram is similar to $\bar{M}_1\bar{M}_2\bar{M}_3\bar{M}_4$, and the sides of the two parallelograms are perpendicular and meet on the diagonals of the quadrilateral $A_1A_2A_3A_4$, particularly at the midpoints of the diagonal segments XA_i

Corollary 2. *Let $A_1A_2A_3A_4$ be a quadrilateral, and X the intersection of its diagonals. Consider the four circles $\bar{K}_i$ through A_i, A_{i+1}, and X. Then the centers $\bar{M}_i$ of the four circles form a parallelogram which is similar to the Varignon parallelogram (see Fig. 10). The sides of the Varignon parallelogram and of the parallelogram $\bar{M}_1\bar{M}_2\bar{M}_3\bar{M}_4$ are perpendicular to each other and meet on the diagonals of the quadrilateral.*

Proof. The lines $\bar{M}_i\bar{M}_{i+1}$ are obviously the perpendicular bisectors of the line segments XA_{i+1}. In particular, $\bar{M}_i\bar{M}_{i+1}$ intersects the segment XA_{i+1} at its midpoint. On the other hand, the sides of the Varignon parallelogram are parallel to the diagonals of the quadrilateral. Hence, the sides of both parallelograms are perpendicular to each other, and both parallelograms share the same angles. Notice also, that the sides of the Varignon parallelogram are the midlines of the triangles consisting of a diagonal and two adjacent quadrilateral sides. In particular, the sides of the Varignon parallelogram intersect the segments XA_{i+1} also at their midpoint. Moreover, it follows that the ratio of the sides of the Varignon parallelogram equals the ratio of the diagonals of the quadrilateral. On the other hand, the altitudes of the parallelogram $\bar{M}_1\bar{M}_2\bar{M}_3\bar{M}_3$, and hence the sides, have the same ratio. It follows that the Varignon parallelogram is similar to the parallelogram $\bar{M}_1\bar{M}_2\bar{M}_3\bar{M}_3$. □

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