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The Inverse Caustic Problem

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Abstract. For a family of parallel incoming light rays, we consider the problem of finding a reflecting curve Γ that has as its caustic a given curve γ. First, a geometric string construction for Γ is presented. Then the differential equation of the curve Γ is derived and its analytical properties are discussed. We conclude with two concrete examples, and a remark about the cusps of a caustic.

1. INTRODUCTION. Suppose a family of light rays in the plane is reflected at a curve or refracted by an object. The envelope of the reflected rays is called the *catacaustic*, or just *caustic*, of the curve, and the *diacaustic* in the case of refracted rays. Usually the generating family either consists of parallel light rays or light rays that emanate from a point. One of the most prominent examples is the nephroid which occurs as the caustic of the (half-)circle and which can be observed on a sunny day at the surface of a cup of coffee (see Figure 1).

Figure 1. Caustic of a circle.

The notion of the caustic comes up in a letter [**15**, p. 484] from Ehrenfried Walter von Tschirnhaus to Gottfried Wilhelm Leibniz dated April 7, 1681. In his letter, Tschirnhaus refers to Christiaan Huygens, who described the phenomenon in [**10**]. The subsequent correspondence [**15**, pp. 491ff] of the two shows Leibniz's high interest in this topic. Tschirnhaus published his results on the theory of caustics in [**17**]. The problem of the caustic soon attracted the most notable mathematicians of the time: Important progress was made by Johann Bernoulli who brought the subject to a certain perfection and who coined the term *Caustica*. The *index rerum* of his *Opera Omnia* [**2**] shows a long list of items under the key word *Caustica*. His brother, Jacob Bernoulli, also treats the subject extensively (see his *Opera* [**1**, pp. 473ff and 549ff]). Guillaume François Antoine, Marquis de L'Hôpital presented in 1693, the year when he became a member of the Académie des sciences, an essay on caustics [**12**, pp. 380–383] and dedicated two chapters in the first French textbook on calculus [**13**, Sec. VI, VII] to the topic. Cayley condensed and extended the theory of caustics in two remarkable essays [**5**, **6**]. For more recent work on caustics, see, e.g., [**3**, **4**, **11**].

Astonishingly, the inverse question, i.e., given a caustic γ, find a reflecting curve Γ that generates the caustic, was not treated until 1859, when Georg Wilhelm Strauch

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investigated the inverse problem in a comprehensive essay [16]. Strauch's idea was to consider the two-parameter family of parabolas with axis parallel to the direction of the generating light rays and focus in the points of the given curve γ. He then finds a reflecting curve Γ as a curve that touches the parabolas to second order. His reasoning, however, is rather heuristic, and unfortunately the resulting solution appears in implicit form as the solution to a system of nonlinear equations, also involving an integration. It is the aim of this article to give a geometric and an explicit analytic solution to the inverse caustic problem. We concentrate on the problem with parallel light rays.

2. THE GEOMETRIC POINT OF VIEW. Let v be a vector in the plane that defines the direction of the incoming light rays, and Γ a smooth curve that is concave with respect to v, with nonvanishing curvature, as in Figure 2.

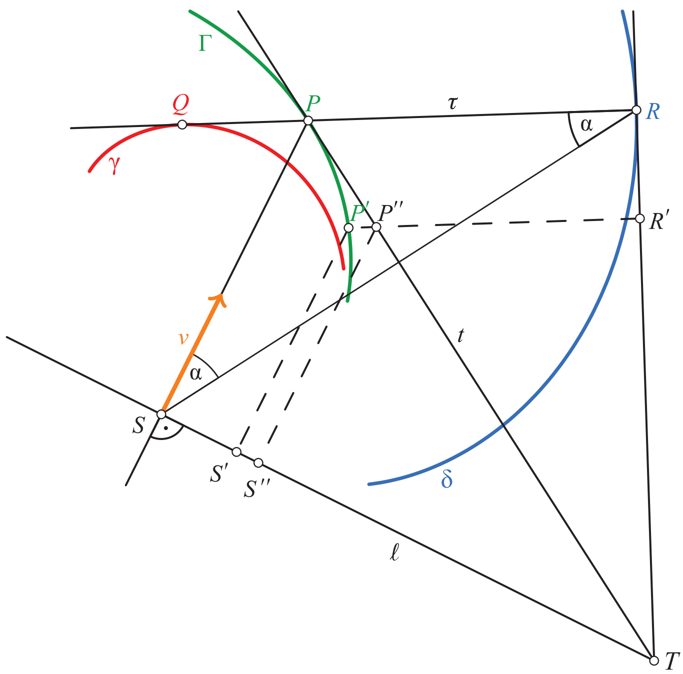

Figure 2. The curve Γ with its caustic γ and its anticaustic δ.

We fix a line ℓ perpendicular to v, a point P on Γ, and we pick the light ray arriving at P. This ray intersects ℓ in a point S, and is reflected in the point P at the tangent t of Γ. The reflected ray τ is tangent to the caustic γ at a point Q. If we reflect S with respect to t, we get the point R. If P varies on Γ, then S runs along ℓ and R runs along the so-called *anticaustic* δ. Hence, the anticaustic is the locus of all points R with the property that

$$|PS| = |PR|.$$

Now, consider the line TR in Figure 2 that is obtained by reflecting ℓ with respect to t. Observe that, for any point $R' \neq R$ on TR, we have

$$|P'S'| < |P''S''| = |P''R'| < |P'R'|.$$

Hence, R' cannot be a point of δ, and δ runs on one side of the line RT. This means that RT is tangent to δ at R. But this implies that δ is an involute of γ. This observation, which goes back to Joseph Diez Gergonne [9, Théorème I], allows us to construct a reflecting curve Γ when the caustic γ is given (see Theorem 1). Γ will be called a *procaustic* of γ.

There is a second way to look at the geometric situation: A circle with center P and radius $|PS|$ is tangent to the line RT at R, and hence also tangent to the anticaustic δ. Hence, δ can also be interpreted as the envelope of circles centered at the curve Γ and tangent to ℓ. This means that δ is an *orthotomic* of Γ, sometimes also called a *secondary caustic* of Γ. Hence the caustic γ is the evolute of an orthotomic of the mirror curve Γ. Or the other way round, the mirror curve Γ is the antiorthotomic of an involute of the caustic γ. More precisely we have the following.

Theorem 1. (i) *Let γ be a smooth curve with nonvanishing curvature and υ a vector indicating the direction of the incoming light rays. Let P be a given point not on γ. Then the number of procaustics of γ through P equals the number of tangents from P to γ.*

(ii) *Each procaustic through P is geometrically characterized as follows: Let τ be a tangent from P to γ with point of tangency Q, and let δ be an involute of γ intersecting τ orthogonally at R such that P lies between Q and R. Finally, let PRS be the isosceles triangle with base SR and SP the direction of υ, and ℓ the perpendicular to υ through S. Then the procaustic Γ of γ through P is the locus of all points P' that are the center of a circle touching both ℓ and δ. (If R and the contact point with ℓ lie on the same side of ℓ, then choose the circle touching δ from inside, else the other one.)*

Remarks.

(a) If τ_1 and τ_2 are two tangents from P to γ that meet at an angle α, then the two procaustics through P meet at an angle $\alpha/2$.

(b) The assumption that the curvature of γ does not vanish can be relaxed. The proof goes through if γ is locally strictly convex.

(c) The procaustic may generate only a part of the curve γ: see Theorem 3.

Proof. The situation of the center P' of a circle touching both ℓ and δ is indicated in Figure 3: The normal τ' in R' to δ through P' is tangent to γ at Q', and $|P'S'| = |P'R'|$, where S' is the foot of the perpendicular from P' to ℓ. Let T' denote the intersection of ℓ and the tangent at R' to δ.

We first show that the line $T'P'$ is tangent to the locus Γ of all points P' with the property $|P'S'| = |P'R'|$. To see this, let X be a point on $T'P'$ different from P'. The normal to δ through X intersects δ in V and touches γ at W. The distance from X to ℓ is $|XY| = |XZ|$, and we want to show that

$$|XV| < |XY|. \tag{1}$$

Indeed, this implies that Γ runs on one side of the line $T'P'$ with P' the only common point and hence, that $T'P'$ is the tangent of Γ at P'. For (1), observe that the following balls are contained in each other:

$$B_{|XV|}(X) \subset B_{|WV|}(W) \subset B_{|Q'R'|}(Q').$$

The first inclusion is trivial. For the second, suppose that $U \in B_{|WV|}(W)$. Then

$$|Q'U| \le |Q'W| + |WU| \le |\widehat{Q'W}| + |WU| \le |\widehat{Q'W}| + |WV| = |Q'R'|,$$

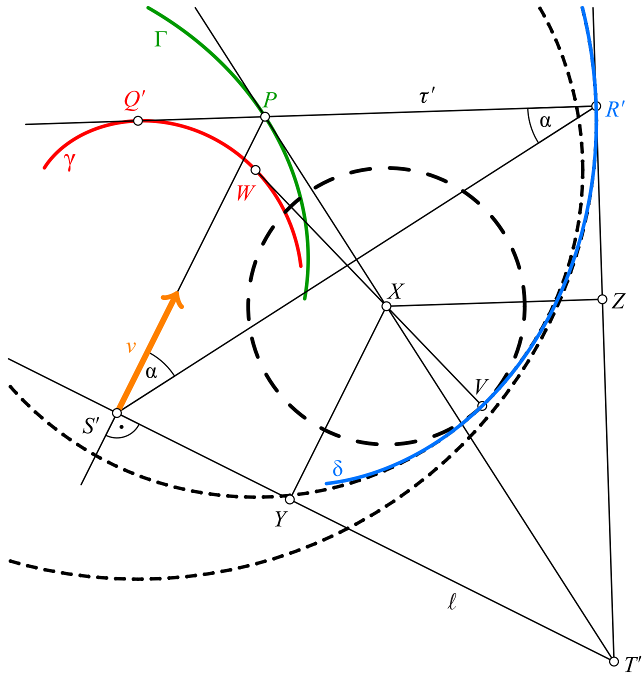

Figure 3. Proof of Theorem 1.

where $\widehat{Q'W}$ is the segment of γ between Q' and W. Hence, since Z lies outside $B_{|Q'R'|}(Q')$, Z lies also outside $B_{|XV|}(X)$, and (1) follows.

Now, by construction, we have $\sphericalangle Q'P'S' = 2\sphericalangle P'R'S'$, and the tangent $P'T'$ is the angle bisector of $\sphericalangle S'P'R'$. This means that the incoming ray $S'P'$, reflected at Γ at the point P', becomes the outgoing ray $P'Q'$ which is tangent to γ. Hence γ is the caustic of Γ, or equivalently, Γ is a procaustic of γ. ■

The previous theorem suggests an easy geometric construction of the procaustic which is shown in Figure 4.

Proposition 2. *A T-square is slid along the line ℓ while a cord is slung around γ and fixed in the corner S' of the T-square. The cord is stretched tight by a pencil in P' sliding along the T-square. Then the pencil traces a procaustic Γ.*

Proof. Let P be a point on the procaustic Γ such that $|PS| = |PR|$, where R is the corresponding point on the anticaustic δ and S is the foot of the perpendicular from P to ℓ (see Figure 4). Suppose the cord from Q over P to S is stretched. We have

$$|QR| = |QP| + |PS|. \tag{2}$$

Now, let P' be a point constructed according to the proposition, i.e.,

$$|QP| + |PS| = |\widehat{QQ'}| + |Q'P'| + |P'S'|. \tag{3}$$

If R' denotes the point where the normal to δ through P' meets δ, we have

$$|\widehat{QQ'}| + |Q'R'| = |QR|. \tag{4}$$

Adding (2), (3), and (4) yields $|Q'R'| = |Q'P'| + |P'S'|$ and hence $|P'R'| = |P'S'|$. Thus P' lies on the procaustic Γ. ■

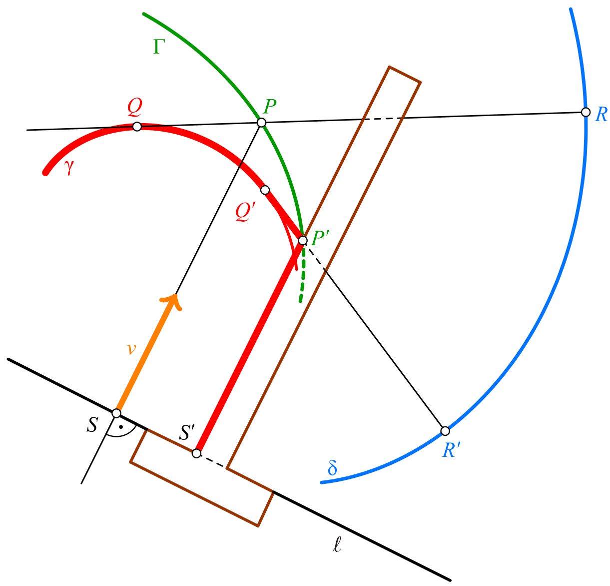

Figure 4. Construction of the procaustic.

Remark. If the generating light rays emanate from a single point S rather than being parallel to a direction v, a corresponding cord construction of the procaustic has been described in [**18**].

3. THE ANALYTIC POINT OF VIEW. Suppose the reflecting curve is twice continuously differentiable, given in parametrized form $\Gamma : (a, b) \to \mathbb{R}^2$, $t \mapsto \Gamma(t)$, and immersed. We assume that Γ is concave with respect to the direction v of incoming light rays and at no point perpendicular to v, where $\|v\| = 1$. We consider an incoming light ray that hits the reflecting curve Γ at $\Gamma(t)$, such that the reflected ray touches the caustic γ at $\gamma(t)$ (see Figure 5).

Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that the geometric condition for the caustic translates into the formulas

$$\langle A\dot{\gamma}(t), \Gamma(t) - \gamma(t)\rangle = 0 \tag{5}$$

$$\dot{\Gamma}(t) = x(t)(v\|\dot{\gamma}(t)\| + \dot{\gamma}(t)) \tag{6}$$

where x is a scalar function, and where we assume that $\dot{\gamma}(t) \neq 0$. Equation (5) indicates that the line through $\gamma(t)$ and $\Gamma(t)$ is tangent to γ at the point $\gamma(t)$, and (6) means that the angle of incidence equals the angle of reflection, where we assume that the parametrization of Γ is such that $\dot{\gamma}(t)$ is pointing in the direction of the corresponding reflected ray. By differentiating (5) we get

$$0 = \langle A\ddot{\gamma}, \Gamma - \gamma\rangle + \langle A\dot{\gamma}, \dot{\Gamma}\rangle. \tag{7}$$

Now we replace $\dot{\Gamma}$ in (7) by (6) and solve for x to arrive at

$$x = \frac{\langle A\ddot{\gamma}, \gamma - \Gamma\rangle}{\langle A\dot{\gamma}, v\|\dot{\gamma}\|\rangle}.$$

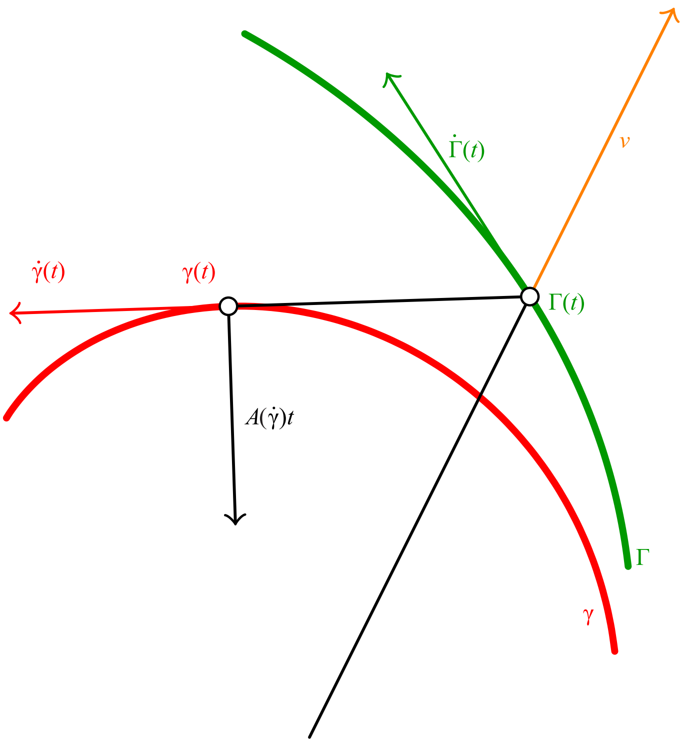

Figure 5. Caustic γ and reflecting curve Γ.

Observe that the denominator is not zero by the assumption that Γ is nowhere perpendicular to v. Using this expression for x in (6) we find that the reflecting curve Γ necessarily solves the system of coupled differential equations

$$\dot\Gamma = \frac{\langle A\ddot\gamma, \gamma - \Gamma\rangle}{\langle A\dot\gamma, v\rangle}\left(v + \frac{\dot\gamma}{\|\dot\gamma\|}\right),$$

provided $\dot\gamma$ does not vanish. Vice versa, we have the following.

Theorem 3. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be an immersed curve of class C^2 such that $\dot\gamma$ is at no point parallel to the direction v of the incoming light rays, $\|v\| = 1$, and $\ddot\gamma$ does not vanish. Assume that for a point $t_0 \in (a, b)$ and some $\lambda > 0$ we have $\Gamma_0 = \gamma(t_0) - \lambda\dot\gamma(t_0)$. Then there exists a unique solution Γ of*

$$\dot\Gamma = \frac{\langle A\ddot\gamma, \gamma - \Gamma\rangle}{\langle A\dot\gamma, v\rangle}\left(v + \frac{\dot\gamma}{\|\dot\gamma\|}\right) \tag{8}$$

on (a, b) with initial condition $\Gamma(t_0) = \Gamma_0$. Moreover, there is a maximal interval $(a_0, b) \subset (a, b)$ containing t_0 such that $\dot\Gamma \neq 0$ on (a_0, b). Then $\Gamma|_{(a_0,b)}$ is the procausitc of $\gamma|_{(a_0,b)}$ through the point Γ_0 with respect to the direction v of the light. If $a_0 \in (a, b)$, then Γ meets γ at $\gamma(a_0)$ such that it bisects the angle between $\dot\gamma(a_0)$ and v.

Proof. The nonhomogeneous linear system (8) of differential equations can be written in the form $\dot\Gamma(t) = U(t)\Gamma(t) + V(t)$ where $U : (a, b) \to \mathbb{R}^{2\times 2}$ and $V : (a, b) \to \mathbb{R}^2$ are continuous functions with values in real 2×2-matrices and in $\mathbb{R}^2$, respectively. Then, by the standard theory of differential equations there exists a unique solution of (8) with initial value $\Gamma(t_0) = \Gamma_0$ (see, e.g., [7, Theorem 7.4]). Observe that the denominator $\langle A\dot\gamma, v\rangle \neq 0$ on (a, b). On the maximal interval $I \subset (a, b)$, $t_0 \in I$, where $\dot\Gamma$ does not vanish, Γ is immersed.

By taking the inner product of (8) with $A\dot\gamma$ we obtain

$$\langle A\dot\gamma, \dot\Gamma\rangle = \left\langle A\dot\gamma, \frac{\langle A\ddot\gamma, \gamma - \Gamma\rangle}{\langle A\dot\gamma, v\rangle}\left(v + \frac{\dot\gamma}{\|\dot\gamma\|}\right)\right\rangle = \langle A\ddot\gamma, \gamma - \Gamma\rangle,$$

and hence

$$0 = \langle A\ddot\gamma, \Gamma - \gamma\rangle + \langle A\dot\gamma, \dot\Gamma\rangle = \frac{d}{dt}\langle A\dot\gamma, \Gamma - \gamma\rangle.$$

In particular, $\langle A\dot\gamma, \Gamma - \gamma\rangle$ is constant, namely 0, since this is the value of this expression at $t = t_0$. This means geometrically that, if $\Gamma(t) \neq \gamma(t)$, then $\Gamma(t) - \gamma(t)$ is tangent to the curve γ with point of tangency $\gamma(t)$. In particular,

$$\Gamma(t) = \gamma(t) - \lambda(t)\dot\gamma(t) \tag{9}$$

for some function λ with $\lambda(t_0) > 0$ by assumption. But then $\lambda(t) > 0$ on I, because otherwise, by (8), $\dot\Gamma$ would vanish at some point of I. Then, for $t \in I$, it follows directly from (8) that $\dot\Gamma(t)$ is the bisector of the angle between v and $\gamma(t) - \Gamma(t)$. That is, the incoming light ray, reflected with respect to the tangent to Γ at $\Gamma(t)$, becomes the line $\gamma(t) - \Gamma(t)$. Hence $\Gamma|_I$ is the procaustic of γ on I.

Since $\lambda > 0$ on I and $\ddot\gamma \neq 0$, the curves $\gamma(I)$ and $\Gamma(I)$ have no common points. On the other hand, if $a_0 \in (a, b)$ is a boundary point of I, then we have $\dot\Gamma(a_0) = 0$ by continuity. Hence, since $v + \frac{\dot\gamma}{\|\dot\gamma\|} \neq 0$, it follows from (8) that $\gamma(a_0) = \Gamma(a_0)$. Then, from (8) and by using l'Hôpital's rule, we obtain

$$\lim_{t\to a_0} \frac{\dot\Gamma(t)}{t} = \frac{\langle A\ddot\gamma(a_0), \dot\gamma(a_0)\rangle}{\langle A\dot\gamma(a_0), v\rangle}\left(v + \frac{\dot\gamma(a_0)}{\|\dot\gamma(a_0)\|}\right)$$

which shows that Γ meets γ in $\gamma(a_0)$ such that it bisects the angle between $\dot\gamma(a_0)$ and v. Now assume that a_0 is a right endpoint of I. Then, since $\lambda > 0$ on I, we have by (9) that the curve $\Gamma|_I$ is contained in the set

$$\{x = \gamma(t) - \lambda\dot\gamma(t) \mid t \in I, \lambda \geq 0\}.$$

Hence, Γ would meet γ in $\gamma(a_0)$ tangentially, which contradicts the previous finding. Hence, a_0 can only be a left endpoint of I. ■

Remarks.

(a) Theorem 1 allows us to compute an explicit solution of (8) with initial condition $\Gamma(0) = \Gamma_0 := \gamma(0) - \lambda\dot\gamma(0)/\|\dot\gamma(0)\|$. In fact, if $s(t) = \int_0^t \|\dot\gamma(\tau)\| d\tau$ denotes the arclength function of γ, we find

$$\Gamma(t) = \gamma(t) - \dot\gamma(t)\frac{\langle \gamma(t) - S, v\rangle - s(t)}{\langle \dot\gamma(t), v\rangle - \|\dot\gamma(t)\|}, \tag{10}$$

where $S = \Gamma_0 + \lambda v$.

(b) If γ is parametrized by arclength s we have $\|\dot\gamma\| = 1$, and $\langle \ddot\gamma, A\dot\gamma\rangle =: \kappa$ is the signed curvature of γ. Then, since $\gamma - \Gamma = \lambda\dot\gamma$ on I with $\lambda > 0$, we have $\langle A\ddot\gamma, \gamma - \Gamma\rangle = -\kappa\|\gamma - \Gamma\|$, and equation (8) becomes particularly simple:

$$\dot\Gamma = \frac{\kappa\|\gamma - \Gamma\|(v + \dot\gamma)}{\langle \dot\gamma, Av\rangle}. \tag{11}$$

In this case (10) reduces to

$$\Gamma(s) = \gamma(s) - \dot{\gamma}(s)\frac{\langle \gamma(s) - S, v\rangle - s}{\langle \dot{\gamma}(s), v\rangle - 1}. \tag{12}$$

(c) The curve $\Gamma|_{(a,a_0)}$ is a procaustic of $\gamma|_{(a,a_0)}$ with respect to the direction $-v$ of the light rays.

The system (8) is nice and compact, and easy to analyze, but it is coupled. However, it can be reformulated as a decoupled system. To see this, we solve $\langle A\dot{\gamma}, \Gamma - \gamma\rangle = 0$ for the second component of Γ:

$$\Gamma_2 = \frac{1}{\dot{\gamma}_1}\big(\dot{\gamma}_2\Gamma_1 + \langle A\dot{\gamma}, \gamma\rangle\big). \tag{13}$$

This expression is used to replace Γ_2 on the right-hand side of the first equation in (8), which then turns into

$$\dot{\Gamma}_1 = \frac{(\gamma_1 - \Gamma_1)(\|\dot{\gamma}\|v_1 + \dot{\gamma}_1)\langle A\ddot{\gamma}, \dot{\gamma}\rangle}{\|\dot{\gamma}\|\dot{\gamma}_1\langle A\dot{\gamma}, v\rangle}.$$

This is a linear differential equation for Γ_1 alone, which can be solved independently from Γ_2. Similarly, one can deduce a decoupled differential equation for Γ_2. This leads to the following:

Theorem 4. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be an immersed curve of class C^2 such that $\dot{\gamma}$ is at no point parallel to the direction v of the incoming light rays, $\|v\| = 1$, and $\ddot{\gamma}$ does not vanish. Assume that for a point $t_0 \in (a, b)$ and some $\lambda > 0$ we have $\Gamma_0 = \gamma(t_0) - \lambda\dot{\gamma}(t_0)$. Then there exists a unique C^1 solution $\Gamma = (\Gamma_1, \Gamma_2)$ of*

$$\dot{\Gamma}_1 = \frac{(\gamma_1 - \Gamma_1)(\|\dot{\gamma}\|v_1 + \dot{\gamma}_1)\langle A\ddot{\gamma}, \dot{\gamma}\rangle}{\|\dot{\gamma}\|\dot{\gamma}_1\langle A\dot{\gamma}, v\rangle} \tag{14}$$

$$\dot{\Gamma}_2 = \frac{(\gamma_2 - \Gamma_2)(\|\dot{\gamma}\|v_2 + \dot{\gamma}_2)\langle A\ddot{\gamma}, \dot{\gamma}\rangle}{\|\dot{\gamma}\|\dot{\gamma}_2\langle A\dot{\gamma}, v\rangle} \tag{15}$$

on (a, b) with initial condition $\Gamma(t_0) = \Gamma_0$. Moreover, there is a maximal interval $(a_0, b) \subset (a, b)$ containing t_0 such that $\dot{\Gamma} \neq 0$ on (a_0, b). Then $\Gamma|_{(a_0,b)}$ is the procausitc of $\gamma|_{(a_0,b)}$ through the point Γ_0 with respect to the direction v of the light. If $a_0 \in (a, b)$, then Γ meets γ in $\gamma(a_0)$ such that it bisects the angle between $\dot{\gamma}(a_0)$ and v.

Remarks.

(a) One has to only solve either (14) or (15), since the other component can be found algebraically from (5).

(b) The inhomogeneous, linear, first order differential equations (14) and (15) can both be solved explicitly by integration.

(c) The factor $\langle A\dot{\gamma}, v\rangle$ does not vanish since $\dot{\gamma}$ is not parallel to v by hypothesis.

(d) If γ is parametrized by arclength, the equations (14) and (15) become particularly nice: Recall that $\langle A\ddot{\gamma}, \dot{\gamma}\rangle$ is then minus the signed curvature of γ.

Before we start the proof of Theorem 4, let us point out that (14) and (15) are singular at τ where $\dot{\gamma}_1(\tau) = 0$ or $\dot{\gamma}_2(\tau) = 0$, respectively. In fact, if $\dot{\gamma}_1(\tau) = 0$, then the right-hand side of (14) is not defined. However, a C^1 solution Γ has the property that the right-hand side of (14) extends continuously to τ (and similarly for (15)).

Nonetheless, it is a priori possible that in a singular point multiple solutions branch off and nonuniqueness occurs. We will see, however, that this is not the case here.

Proof of Theorem 4. We have seen that the C^1 solution Γ of (8) has the property that $\langle A\dot{\gamma}, \Gamma - \gamma\rangle = 0$. Hence, (13) is valid as long as $\dot{\gamma}_1(t) \neq 0$ and (14) follows by inserting the expression for Γ_2 from (13) in the first equation of (8). Points τ at which $\dot{\gamma}_1(\tau) = 0$ are isolated since $\ddot{\gamma}$ does not vanish by assumption. Hence, by continuity of $\dot{\Gamma}$, the right-hand side of (14) extends continuously to such a τ. The same argument applies to (15), showing that a solution Γ of (8) solves (14) and (15).

Now, let Λ be a second C^1 solution of (14) and (15) with the same given initial condition in t_0. Then, by the uniqueness of solutions of systems of nonhomogeneous linear differential equations with the same initial value (see again [**7**, Theorem 7.4]), Γ_1 agrees with Λ_1 on a maximal interval I where $\dot{\gamma}_1 \neq 0$. To show that Γ_1 agrees with Λ_1 on the whole interval (a, b), we have to show that the only C^1 solution ω of the homogeneous part of (14) with $\omega(\tau) = \dot{\omega}(\tau) = 0$ vanishes identically. By a parameter shift we may assume that $\tau = 0$. So, let ω be a C^1 solution of

$$\dot{\omega} = -\omega \frac{(\|\dot{\gamma}\| v_1 + \dot{\gamma}_1)\langle A\ddot{\gamma}, \dot{\gamma}\rangle}{\|\dot{\gamma}\|\dot{\gamma}_1\langle A\dot{\gamma}, v\rangle}. \tag{16}$$

Observe first that $\dot{\gamma}_1(0) = 0$ implies $\dot{\gamma}_2(0) \neq 0$, $v_1 \neq 0$, and $\ddot{\gamma}_1(0) \neq 0$. Then we may assume that $\gamma_2(t) = t$ in a neighborhood of $t = 0$ since this can be achieved by a reparametrization. One can check that in this case (16) has the general solution

$$\omega(t) = \frac{c\dot{\gamma}_1(t)(\langle\dot{\gamma}, v\rangle + \|\dot{\gamma}(t)\|)}{\langle A\dot{\gamma}(t), v\rangle^2}, \quad c \in \mathbb{R}.$$

One finds

$$\omega(0) = 0 \quad \text{and} \quad \dot{\omega}(0) = \frac{c(1 + v_2)\ddot{\gamma}_1(0)}{v_1^2}.$$

Thus, the only solution with $\omega(0) = \dot{\omega}(0) = 0$ is $\omega(t) \equiv 0$, and hence $\Gamma_1 \equiv \Lambda_1$ on (a, b). The same arguments yield $\Gamma_2 \equiv \Lambda_2$ on (a, b).

We conclude that the solution Γ from Theorem 3 is the only C^1 solution of (14) and (15). Thus, the remaining assertions follow from Theorem 3. ■

The case of incoming light being parallel to the caustic. In Theorems 3 and 4 we have excluded the situation where the direction v of the light coincides with a tangent of γ. Let us now take a closer look at the two cases where, for some τ, v is parallel to $\dot{\gamma}(\tau)$ on the one hand, and where v is antiparallel to $\dot{\gamma}(\tau)$ on the other hand. For simplicity, we assume that γ is parametrized by arclength.

Theorem 5. *Let $\gamma : (a, b) \to \mathbb{R}^2$ be of class C^2, parametrized by arclength, such that $\ddot{\gamma}$ does not vanish, and suppose that $\dot{\gamma}(\tau) = v$ for one value $\tau \in (a, b)$, where v is the direction of the incoming light rays.*

- *Assume that for a point $t_0 \in (a, \tau)$ and some $\lambda > 0$ we have $\Gamma_0 = \gamma(t_0) - \lambda\dot{\gamma}(t_0)$. Then there exists a unique solution Γ of (8) on (a, τ) with initial condition $\Gamma(t_0) = \Gamma_0$. Moreover, there is a maximal interval $(a_0, \tau) \subset (a, \tau)$ containing t_0 such that $\dot{\Gamma} \neq 0$ on (a_0, τ). Then $\Gamma|_{(a_0,\tau)}$ is the procausitc of $\gamma|_{(a_0,\tau)}$ through the point Γ_0 with respect to the direction v of the light and $\|\Gamma(t)\| \to \infty$ for $t \nearrow \tau$.*
- *Assume that for a point $t_0 \in (\tau, b)$ and some $\lambda > 0$ we have $\Gamma_0 = \gamma(t_0) - \lambda\dot{\gamma}(t_0)$. Then there exists a unique solution Γ of (8) on (τ, b) with initial condition*

$\Gamma(t_0) = \Gamma_0$. *Moreover, there is a maximal interval* $(a_0, b) \subset (\tau, b)$ *containing* t_0 *such that* $\dot{\Gamma} \neq 0$ *on* (a_0, b). *Then* $\Gamma|_{(a_0,b)}$ *is the procausitc of* $\gamma|_{(a_0,b)}$ *through the point* Γ_0 *with respect to the direction* v *of the light. If* $a_0 = \tau$, *then either* $\|\Gamma(t)\| \to \infty$ *or* $\Gamma(t) \to \gamma(t)$ *for* $t \searrow \tau$.

Proof. This follows from Theorem 3 and (11). ■

Theorem 6. *Let* $\gamma : (a, b) \to \mathbb{R}^2$ *be of class* C^2, *parametrized by arclength, such that* $\ddot{\gamma}$ *does not vanish, and suppose that* $\dot{\gamma}(\tau) = -v$ *for one value* $\tau \in (a, b)$, *where* v *is the direction of the incoming light rays. Assume that for a point* $t_0 \in (a, b)$ *and some* $\lambda > 0$ *we have* $\Gamma_0 = \gamma(t_0) - \lambda\dot{\gamma}(t_0)$. *Then there exists a unique solution* Γ *of (8) on* (a, b) *with initial condition* $\Gamma(t_0) = \Gamma_0$. *Moreover, there is a maximal interval* $(a_0, b) \subset (a, b)$ *containing* t_0 *such that* $\dot{\Gamma} \neq 0$ *on* (a_0, b). *Then* $\Gamma|_{(a_0,b)}$ *is the procausitc of* $\gamma|_{(a_0,b)}$ *through the point* Γ_0 *with respect to the direction* v *of the light.*

Proof. In this case, the right-hand side of (8) extends continuously to τ. Indeed, by l'Hôpital's rule, we have

$$\lim_{t\to\tau} \frac{v + \dot{\gamma}}{\langle A\dot{\gamma}, v\rangle} = \frac{\ddot{\gamma}(\tau)}{\langle A\ddot{\gamma}(\tau), v\rangle}.$$

Observe that $\langle A\ddot{\gamma}(\tau), v\rangle \neq 0$. The rest of the proof is identical to the proof of Theorem 3. ■

Examples. Let us have a look at the case of the unit circle $\gamma : [0, 2\pi] \to \mathbb{R}^2, t \mapsto \big(\cos(t), \sin(t)\big)$. Here, for $v = (0, 1)$, we obtain by (12)

$$\Gamma(t) = \left((t - 2c)\cot\left(\frac{t}{2}\right) - 1,\ \frac{1}{2}\cos(t)\csc\left(\frac{t}{2}\right)^2\big(2c - t + \tan(t)\big)\right),$$

for arbitrary $c \in \mathbb{R}$. Figure 6 shows the procaustics Γ for some values of c. The solid curves are caustics of the unit circle on the interval $(0, 2\pi)$. For $\tau = 0$ we have $\dot{\gamma}(\tau) = v$: The dashed curve realizes the solution with $\Gamma(t) \to \gamma(t)$ as $t \searrow 0$, whereas the solid curves diverge for $t \searrow 0$ (see Theorem 5). The dotted curves Γ hit the caustic γ at an angle that bisects the angle between v and the tangent vector at γ (see Theorem 3). Observe that for $\tau = \pi$, we have $\dot{\gamma}(\tau) = -v$: The procaustics are smooth curves in $\tau = \pi$ (see Theorem 6). All procaustics diverge for $t \nearrow 2\pi$ (see Theorem 5).

In the second example, the caustic

$$\gamma : (0, \pi) \to \mathbb{R}^2, t \mapsto \left(\cos(t)^3, \sin(t)\left(1 + \frac{1}{2}\cos(2t)\right)\right)$$

is the nephroid. Of course we know one of its procaustics, namely the semicircle, but there is a whole family of procaustics. Since the nephroid has a cusp singularity at $t = \pi/2$, we have to apply Theorems 5 and 6 on both halves of the interval $(0, \pi)$ separately. For $v = (0, 1)$ one finds by (10)

$$\Gamma(t) = \big(\cos(t) - 2c\cot(t), c(\cot(t)^2 - 1) + \sin(t)\big),$$

for arbitrary $c \in \mathbb{R}$. Figure 7 shows the procaustics Γ for some values of c. The solid curves are defined on the interval $(0, \pi)$. The dashed curve is the semicircle, obtained for $c = 0$.

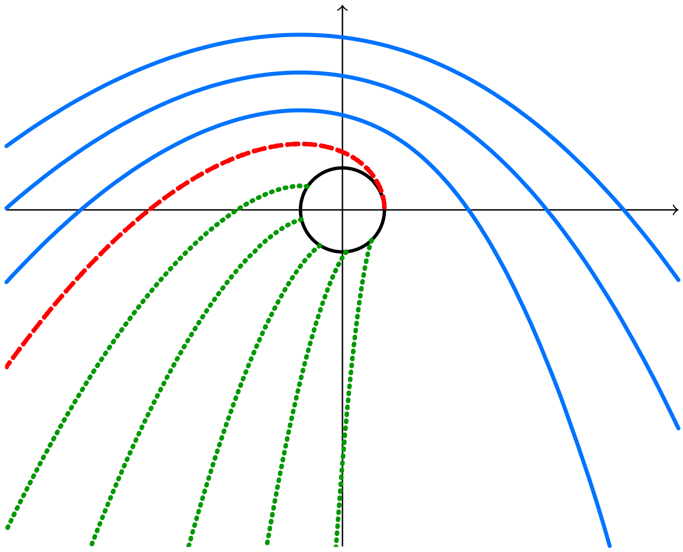

Figure 6. Procaustics of the circle.

Cusps. Even if the mirror curve Γ is smooth it is a common phenomenon that its caustic γ exhibits cusps. Formally these are points on γ where the left and the right limits of the tangent vector are antiparallel. Looking at the two examples in Figures 6 and 7 it is interesting to see that the solid mirror curves in the first example yield a smooth caustic, the unit circle, while in the second example a cusp occurs in the nephroid. Why is that—when do we observe cusps?

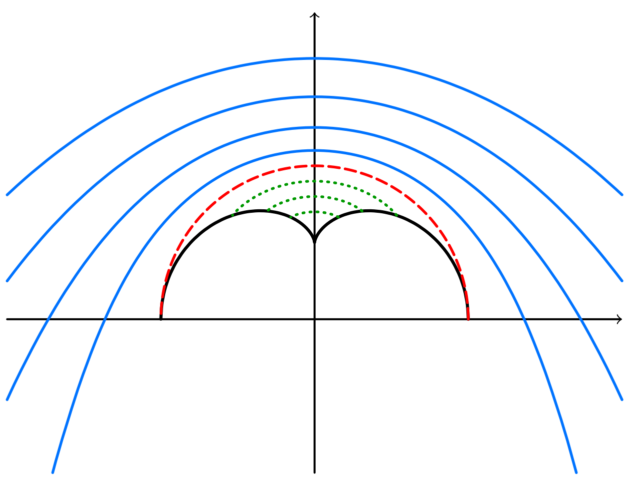

Figure 7. Procaustics of the nephroid.

The occurrence of cusps is readily explained in light of the geometric discussion in Section 2: Recall that the caustic γ is the evolute of the orthotomic δ of the mirror curve Γ. In 1919 George Heyser Light observed in [**14**] that an evolute has a cusp whenever the curvature of its generating curve has a strict local extremum (see, e.g., [**8**, Lemma 10.1] for a modern presentation). So, in our case, the local extrema of the curvature of the orthotomic curve δ are responsible for the cusps of the caustic γ. The curvature of δ can be expressed in terms of Γ, and therefore the question of the appearance of cusps of the caustic can be answered directly by looking at the mirror curve. The result can be formulated particularly nicely if we assume that Γ is given as the graph of a concave function $G : (a, b) \to \mathbb{R}^+$, that the direction of light is $v = (0, 1)$, and that ℓ is the x-axis (see Figure 8).

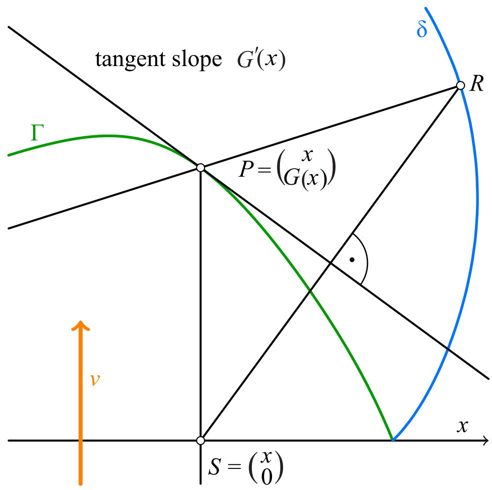

Figure 8. Orthotomic of the mirror curve Γ.

We compute the coordinates of the point R and therefore the parametrization of the orthotomic δ. We get

$$\delta(x) = \begin{pmatrix} x \\ 0 \end{pmatrix} + \frac{2G(x)}{1 + G'(x)^2} \begin{pmatrix} G'(x) \\ 1 \end{pmatrix},$$

and for the signed curvature of δ,

$$\kappa(x) = \frac{2G''(x)}{1 + G'(x)^2 - 2G(x)G''(x)}.$$

Since we need to check for local extrema of κ, we finally compute

$$\kappa'(x) = \frac{2G'''(x)\left(1 + G'(x)^2\right)}{\left(1 + G'(x)^2 - 2G(x)G''(x)\right)^2}.$$

With the previously mentioned result of G. H. Light we obtain the following.

Proposition 7. *Let the mirror curve Γ be given by the graph of a concave C^3 function $G : (a, b) \to \mathbb{R}^+$, and the direction of light $v = (0, 1)$. Moreover, let the caustic γ be parametrized such that $\gamma(x) - \Gamma(x)$ is the direction of the light ray that is reflected at the point $\Gamma(x)$. Then γ has a cusp in $\gamma(x_0)$ pointing away from $\Gamma(x_0)$ if G''' changes its sign in x_0 from positive to negative, and a cusp in $\gamma(x_0)$ pointing toward $\Gamma(x_0)$ if G''' changes its sign in x_0 from negative to positive.*

This result is quite nice: Indeed, the zeros of the first derivative of a function indicate local extrema, the zeros of the second derivative indicate inflection points of the graph, but Proposition 7 interprets the zeros of the *third* derivative geometrically.

An example of the result is illustrated in Figure 9.

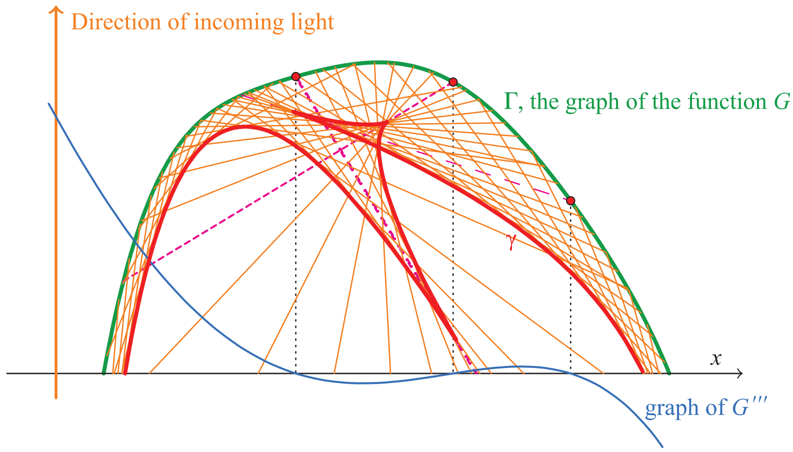

Figure 9. Cusps of the caustic γ of the mirror curve Γ that is the graph of a function G: The three fat dots mark points $(x, G(x))$ where $G'''(x) = 0$. Some reflected light rays are drawn.

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100 Years Ago This Month in *The American Mathematical Monthly*
Edited by Vadim Ponomarenko

At the meeting of the National Research Council on April 28, the Division of Physical Sciences voted to increase its membership by adding a representative of the MATHEMATICAL ASSOCIATION OF AMERICA, and a tenth member-at-large, who should represent mathematics. There were previously in the division fourteen physicists, three representatives of the American Mathematical Society (L. E. DICKSON, O. VEBLEN, and H. S. WHITE), three representatives of the American Astronomical Society, one meteorologist and one geodesist.

A new organization, to be known as the National Council of Teachers of Mathematics, was launched at Cleveland, Ohio, on February 24. About 150 persons were present, representing 20 different states and many different organizations of primary and secondary school teachers in various parts of the country.

Mathematics Teacher, in reorganized form, will probably be the official organ of the Council, administered by an editorial board of from three to five members and an editor in chief. This board is to consist of teachers of elementary and secondary mathematics, and to include a member representing the college group in an advisory capacity.

—Excerpted from “Notes and News” (1920). 27(5): 237–242.