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Research Article

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Solving the quartic by conics

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Abstract: Two conic sections C_1 and C_2 in the Euclidean plane that pass through two given points can generally have two further points of intersection. It is shown how these can be constructed using compass and ruler. The idea is to construct the degenerate conics in the pencil of the two conics C_1 and C_2. Their intersections are then the four intersection points of C_1 and C_2. The same idea is then used to reduce a general quartic equation to a cubic equation and to solve it. This is performed by interpreting the solutions of the quartic as intersections of two complex conic sections. The degenerate complex conics in their pencil can then be found through a cubic equation.

Keywords: intersection of conics, pencils of conics, quartic equations, cubic equations, ruler and compass constructions

MSC 2020: 51M05, 51M15, 12D10

1 Introduction

We investigate the problem of finding the intersection points of two conic sections from a geometric and from an algebraic point of view. For the geometric solution, we work in the Euclidean plane, whereas for the algebraic solution, we work in the complex plane. Let us first have a look at the Euclidean plane where conics are given by quadratic equations in the variables x and y, in the form

$$a_{xx}x^2 + a_{xy}xy + a_{yy}y^2 + a_x x + a_y y + a = 0 \tag{1}$$

with real coefficients. The resultant of two such quadratic equations with respect to x is a quartic equation in y, and the resultant with respect to y is a quartic equation in x. The coefficients of these two quartic equations are polynomials in the coefficients of the two quadratic equations. The solutions of the quartic in x are the x-coordinates of the intersections of the two conics. The corresponding y-coordinates are then the common solutions of the quadratic equations of the two conics for the given values of x. Conversely, we will see that we can interpret the solutions of a quartic equation in the variable x as x-coordinates of the intersections of two conics.

Alternatively, one can also argue as follows by considering the Euclidean plane as embedded in the projective plane. If a point P of one of two conics is known, one can apply a projective transformation to send P to a point at infinity, and we can therefore assume that one conic is a parabola. Note that, in general, the solution of a quadratic equation is required to find such a point P. We may then assume, by a suitable affine transformation, that one conic is the normal parabola $y = x^2$, and the other conic is given by (1). If we then substitute $y = x^2$ into (1), we obtain a quartic equation in x.

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This article is organized as follows. In Section 2, we present a construction of the intersections of two conics by ruler and compass, provided two intersections are known. In Section 3, we use the geometric idea to show how a quartic equation can be reduced to a cubic equation by considering the pencil of two associated conics.

2 Construction of the intersection of two conic sections

As discussed in Section 1, the problem of finding the intersection points of two conic sections corresponds to a quartic equation. It is therefore, in general, impossible to construct the intersections by ruler and compass. However, if two points of intersection are known, the problem reduces to a quadratic equation, and a construction should be feasible. In [1], such a construction was used to intersect a circle and a hyperbola in a special position. In this section, we want to present a construction, which works for two arbitrary conics for which two common points are known. We work in the framework of the real projective plane. The construction with ruler and compass takes place in the Euclidean plane which we consider as embedded in the projective plane. The compass will only be used in one single construction step.

Before we describe the construction, we will prove a lemma concerning quadruples of points forming harmonic ranges. Recall that four points A, B, X, and Y on a line in the real projective plane form a *harmonic range*, denoted $(A, B;\, X, Y)$, if their cross-ratio satisfies $\operatorname{cr}(A, B;\, X, Y) = -1$.

Lemma. *Let C_1 and C_2 be two conics that meet in the four points I_1, I_2, J_1, and J_2* (Figure 1). *Let $r := I_1I_2$, let P_1 and Q_2 be the poles of the polar line r with respect to C_1 and C_2, respectively, let p_2 be the polar line of P_1 with respect to C_2, and let q_1 be the polar line of Q_2 with respect to C_1. Furthermore, let $s := P_1Q_2$; R, T_1, and T_2 be the intersection points of s with r, q_1, and p_2, respectively; and let A_1, B_1 and A_2, B_2 be the intersection points of s with C_1 and C_2, respectively. Finally, let H_1 and H_2 be the intersection points of I_1J_2 with I_2J_1, and I_1J_1 with I_2J_2, respectively.*

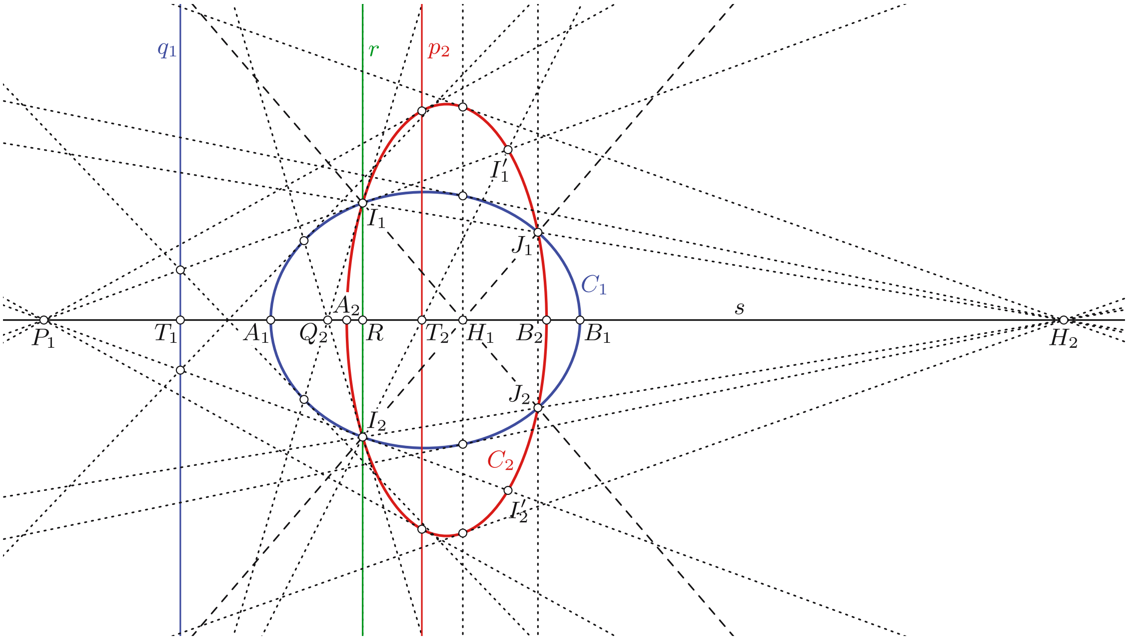

Figure 1: Illustration for the lemma. Using a projective mapping, we can assume that the conic sections C_1 and C_2 are in a symmetrical position.

Then, we have the following:

(*a*) *The points* $A_1, B_1, A_2, B_2, H_1,$ *and* H_2 *are collinear.*
(*b*) $\mathrm{cr}(P_1, R;\, A_1, B_1) = \mathrm{cr}(Q_2, T_1;\, A_1, B_1) = -1$.
(*c*) $\mathrm{cr}(P_1, T_2;\, A_2, B_2) = \mathrm{cr}(Q_2, R;\, A_2, B_2) = -1$.
(*d*) $\mathrm{cr}(A_1, B_1;\, H_1, H_2) = \mathrm{cr}(A_2, B_2;\, H_1, H_2) = -1$.

Proof. By a projective transformation, we can assume that the four points $I_1, I_2, J_1,$ and J_2 form an isosceles trapezoid, where the lines I_1I_2 and J_1J_2 are parallel. In this situation, the entire configuration is mirror symmetric with respect to s. In particular, the lines $r, p_2,$ and q_1 are parallel (as in Figure 1).

(a) follows directly from the symmetry. (b), (c), and (d) follow from the following fact (see, e.g., [2, Section 6·4] or [3, Satz 4.9]): Let C be a conic, let P be a point not on C, let p be the polar line of P with respect to C, and let s be a line through P which intersects p at T and the conic C at the two points A and B. Then, the four points $P, T, A,$ and B on s form a harmonic range $(P, T;\, A, B)$. □

Now, we are ready to present a construction of the intersections of two conics C_1 and C_2 by ruler and compass, provided that the two conics have four intersection points and that two intersections I_1 and I_2 are known.

Construction. Since a conic is determined by five points, assume that C_1 and C_2 are given by the points $I_1, I_2, C_{1,1}, C_{1,2}, C_{1,3}$ and $I_1, I_2, C_{2,1}, C_{2,2}, C_{2,3}$, respectively. Then, we construct the following points and lines.

(1) Point P_1: By Pascal's theorem, we can construct by ruler alone the tangents to the conic C_1 at the points I_1 and I_2 (see Figure 2). The intersection of these two tangents is the point P_1.
(2) Point Q_2: The point Q_2 is constructed as above as the intersection of the tangents in I_1 and I_2 with respect to the conic C_2.
(3) Line s: Joining the points P_1 and Q_2, we obtain the line s.
(4) Point R: Intersecting the line $r = I_1I_2$ with s gives the point R.
(5) Point T_2: By Pascal's theorem we can construct by ruler alone the intersection point I_1' of C_2 with the line P_1I_1 (see Figure 3). Then, T_2 is the intersection of I_2I_1' and s (see Figure 1 and [2, Section 6·4] or [3, Satz 4.10]). Note that I_1', T_2, I_2 as well as I_2', T_2, I_1 are collinear (see Figure 4 for the general situation).
(6) Point T_1: The point T_1 is constructed in the same way as T_2, above, with the point Q_2 in place of P_1 and the conic C_1 in place of C_2.
(7) Points A_1 and B_1: By item (b) of the aforementioned lemma, we have the harmonic ranges $(P_1, R;\, A_1, B_1)$ and $(Q_2, T_1;\, A_1, B_1)$. Hence, the points A_1 and B_1 are determined by the points P_1, R, Q_2, T_1 – which we have already constructed – and can be constructed using a compass and ruler (see Figure 5, and [3, p. 78]).

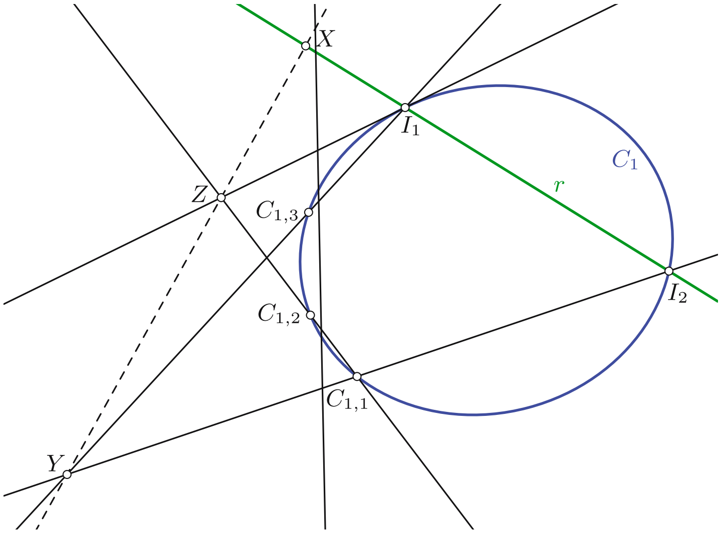

Figure 2: Construction of the tangent in I_1: XY is the Pascal line in the hexagon $I_1\,I_2\,C_{1,1}\,C_{1,2}\,C_{1,3}\,I_1$. Hence, ZI_1 is the tangent to C_1 in I_1.

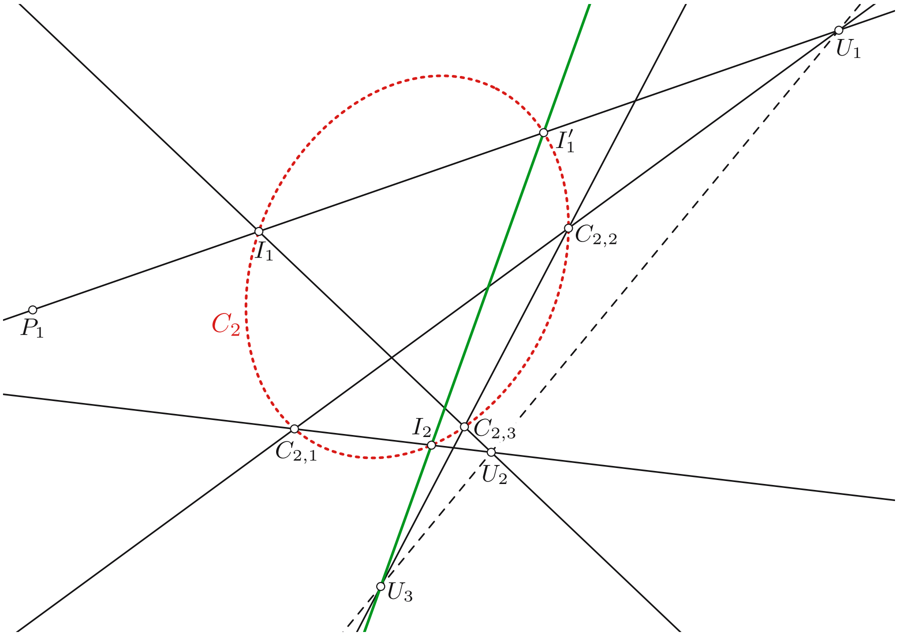

Figure 3: With U_1 and U_2, we can construct the Pascal line of the hexagon $I_1\,I_1'\,I_2\,C_{2,1}\,C_{2,2}\,C_{2,3}$. With the Pascal line and the line through $C_{2,2}$ and $C_{2,3}$, we obtain U_3, and I_1' is the intersection of $U_3\,I_2$ and $P_1\,I_1$.

(8) Points A_2 and B_2: The points A_2 and B_2 are constructed as above with respect to the points P_1, T_2, Q_2, and R and using item (c) of the lemma.

(9) Points H_1 and H_2: The points H_1 and H_2 are constructed as above with respect to the points A_1, B_1, A_2, and B_2 and using item (d) of the lemma.

(10) Points J_1 and J_2: Finally, the other two intersection points J_1 and J_2 of C_1 and C_2 are obtained as follows: J_1 is the intersection of the lines I_1H_2 and I_2H_1, and J_2 is the intersection of the lines I_1H_1 and I_2H_2. Observe that the line pairs $\{I_1\,J_2\,,\,I_2\,J_1\}$ and $\{I_1\,J_1\,,\,I_2\,J_2\}$ are degenerate conics in the pencil generated by C_1 and C_2.

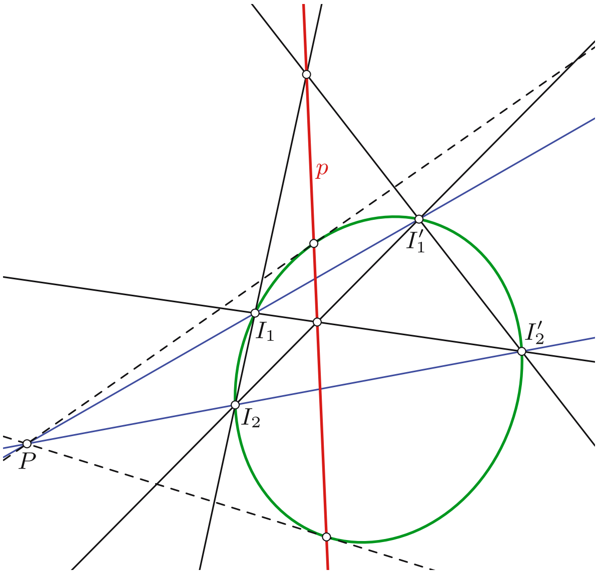

Figure 4: Line p is the polar line with respect to the pole P, and PI_1 and PI_2 are arbitrary secants.

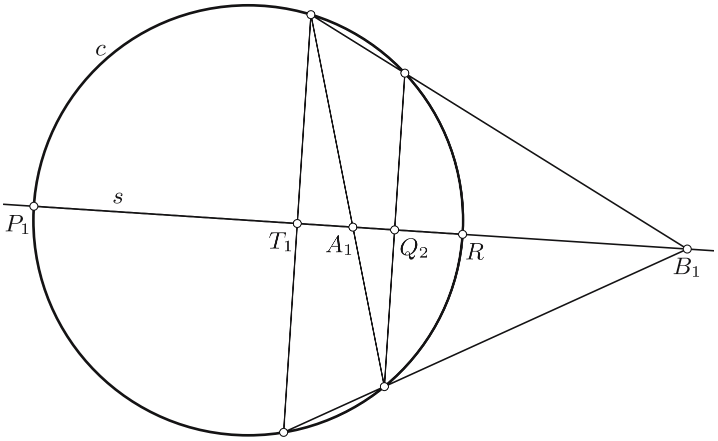

Figure 5: Construction of the points A_1 and B_1. The circle c has P_1R as a diameter. The segments in T_1 and Q_2 are orthogonal to s. $(P_1, R; A_1, B_1)$ is a harmonic range since A_1 is a point of the polar of B_1 with respect to c, and $(Q_2, T_1; A_1, B_1)$ is a harmonic range because the points are part of the complete quadrangle given by the straight lines.

3 Solving the quartic by conics

It is well known that a quartic equation in the variable z can be reduced to a cubic equation. The usual procedure is to first obtain rid of the cubic term in the quartic equation. This is performed by a substitution $z = x - \mu$, leading to the depressed quartic in the variable x. Then, one can follow Ferrari's solution, for example, which leads to a cubic equation (see, e.g., [4, §3.2], or [5] for a full deduction of the formulas). Another way is to employ Galois theory and factorization to reduce the quartic to a cubic problem (see, e.g., [6]). In the course of time, numerous renowned mathematicians have given solution methods for the cubic and quartic equations. Among them are Vieta, Descartes, van Schooten, Euler, Lagrange, Bézout, Lebesgue, and others (see [7] or [8] for an extensive overview of the known solution methods).

In this section, we want to show an alternative way to reduce the quartic to a cubic equation. The idea is inspired by the geometric construction in Section 2. To make this idea work, we will reformulate a given quartic equation as a system of two quadratic equations in two variables that represent two conics C_1 and C_2. Observe that the pairs of straight lines $I_1 J_1, I_2 J_2$, and $I_1 J_2, I_2 J_1$ that we used in the construction are degenerate conics in the pencil spanned by C_1 and C_2. To compute these (in general complex) degenerate conics from the equations of C_1 and C_2 is indeed only a cubic problem. The intersection points of the straight lines, and thus the solutions of the original quartic, can then be calculated immediately. In terms of computational effort, this method is no better or worse than others. But since it is geometrically motivated, it can be visualized particularly well. Before we describe the general case, we illustrate the method with the following example.

Example. Suppose we want to solve the quartic

$$z^4 + z^3 - 45z^2 - 97z + 140 = 0. \tag{2}$$

To have it easier later on, we first remove the quadratic term in the equation. If we substitute $z = x - \mu$, we find the coefficient $3(2\mu^2 - \mu - 15)$ for x^2. This quadratic expression vanishes for $\mu = 3$. The resulting quartic equation in x is then

$$x^4 - 11x^3 + 92x + 80 = 0. \tag{3}$$

Now we consider the two conics

$$C_1 : x^2 - y = 0, \tag{4}$$

$$C_2 : y^2 - 11xy + 92x + 80 = 0. \tag{5}$$

Clearly, if x and y is a solution of the system (4)–(5), then x solves (3), and *vice versa*, if x is a solution of (3), then $x, y = x^2$ is a solution of (4)–(5).

In what follows, it is convenient to consider the Euclidean plane as embedded in the real projective plane by

$$\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}), \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

In this sense, we can write the first conic C_1 as $\langle X, AX \rangle = 0$, where $X = (x, y, 1)^\top$, and A may, for convenience, be taken as

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. Similarly, the second conic C_2 is given by the equation $\langle X, BX \rangle = 0$ with

$$B = \begin{pmatrix} 0 & -11 & 92 \\ -11 & 2 & 0 \\ 92 & 0 & 160 \end{pmatrix}.$$

The pencil of the two conics, i.e., the conics whose equations are linear combinations of (4) and (5), is the set of conics given by equations of the form

$$\langle X, (\lambda A + \mu B)X \rangle = 0,$$

for scalars λ and μ, not both zero. Observe that all conics of the pencil pass through the intersections of C_1 and C_2. The idea is now to find the values of λ, and μ, not both zero, such that the matrix $\lambda A + \mu B$ is singular, and hence, the corresponding conics of the pencil degenerate to straight lines. Then, the intersections of the conics C_1 and C_2 can simply be computed by intersecting these lines, as indicated in Figure 6. In this case and later, A is nonsingular, so μ cannot be zero; it may be taken as 1, and λ may be required to satisfy $\det(\lambda A + B) = 0$.

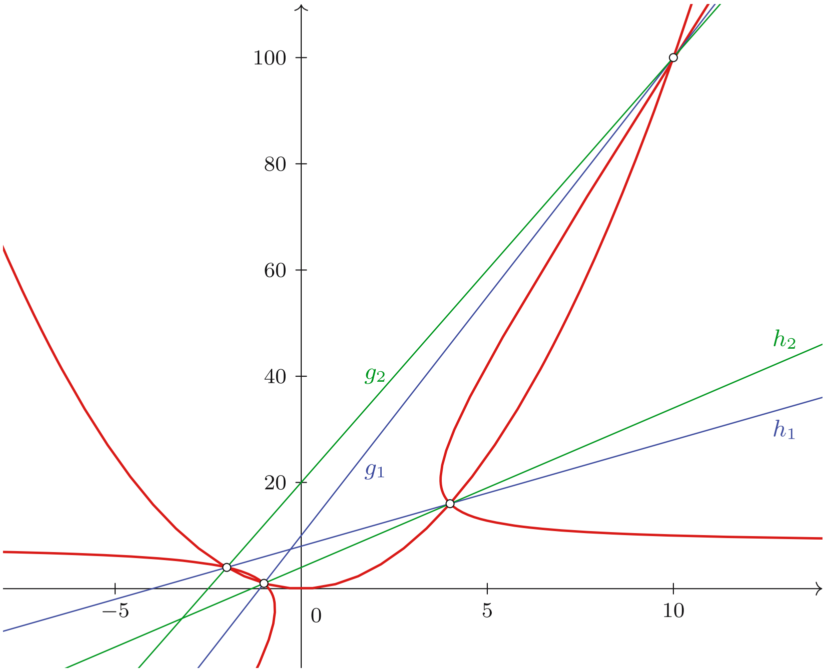

Figure 6: The parabola C_1 and the hyperbola C_2 are shown in red. The degenerate conics in the pencil of C_1 and C_2 are the blue lines g_1 and h_1 and the green lines g_2 and h_2.

The point is that this is a *cubic* equation for λ, which is in our example $\det(\lambda A + B) = -2\lambda^3 + 2664\lambda - 36288 = 0$. Observe that there is no quadratic term, which makes it quite easy to find the roots in general. In our model case, we find the values $\lambda_1 = 18$ and $\lambda_2 = 24$. For these values, the quadratic form

$$\langle X, (\lambda A + B)X\rangle$$

factors in two straight lines as follows (see below for the general case):

$$\langle X, (18A + B)X\rangle = \langle X, g_1\rangle\langle X, h_1\rangle, \quad \langle X, (24A + B)X\rangle = \langle X, g_2\rangle\langle X, h_2\rangle,$$

with

$$g_1 = \begin{pmatrix} 9 \\ -1 \\ 10 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 4 \\ -2 \\ 16 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 8 \\ -1 \\ 20 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 6 \\ -2 \\ 8 \end{pmatrix}.$$

The intersections of these lines can be computed by the respective cross-products. For readers who are less familiar with projective geometry, this can be explained briefly as follows: let, for example, g_1 and g_2 be normal vectors for planes through the origin in $\mathbb{R}^3$ that represent the projective lines. Then, the common point of the projective lines is represented by the intersection of those planes, which is the line from the origin in $\mathbb{R}^3$ perpendicular to both those normals, i.e., in the direction of their cross-product. Formally, this can also be seen by just noting that $X = g_1 \times g_2$ is a solution of the equations $\langle X, g_1\rangle = 0$ and $\langle X, g_2\rangle = 0$ of the two lines. In our example, we find

$$g_1 \times g_2 = -\begin{pmatrix} 10 \\ 100 \\ 1 \end{pmatrix}, \quad g_1 \times h_2 = -12\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad h_1 \times g_2 = 12\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}, \quad h_1 \times h_2 = 4\begin{pmatrix} 4 \\ 16 \\ 1 \end{pmatrix}.$$

We read off the solutions $x_1 = 10$, $x_2 = -1$, $x_3 = -2$, and $x_4 = 4$ of equation (3). Hence, the solutions of the original equation (2) are $z_1 = 7$, $z_2 = -4$, $z_3 = -5$, and $z_4 = 1$.

It is instructive to compare this geometrically motivated method with Ferrari's classical solution. He first removes the cubic term by substituting $z = x - \mu$. The coefficient of x^3 is then $1 - 4\mu$. Hence, by choosing $\mu = \frac{1}{4}$, we obtain the depressed equation

$$x^4 - \frac{363}{8}x^2 - \frac{595}{8}x + \frac{41325}{256} = 0.$$

Ferrari rewrites this as

$$x^4 - 2 \cdot \frac{363}{8}x^2 + \left(\frac{363}{8}\right)^2 = -\frac{363}{8}x^2 + \left(\frac{363}{8}\right)^2 + \frac{595}{8}x - \frac{41325}{256}$$

such that the left-hand side is a square. To make also the right-hand side a square, Ferrari introduces the variable y to obtain

$$\left(x^2 - \frac{363}{8} + y\right)^2 = \left(-\frac{363}{8} + 2y\right)x^2 + \frac{595}{8}x + \left(\frac{363}{8}\right)^2 - \frac{41325}{256} - 2 \cdot \frac{363}{8}y + y^2. \tag{6}$$

The right-hand side is a square, if the discriminant of the quadratic expression vanishes, i.e., if

$$\left(\frac{595}{8}\right)^2 = 4\left(-\frac{363}{8} + 2y\right)\left(\left(\frac{363}{8}\right)^2 - \frac{41325}{256} - 2 \cdot \frac{363}{8}y + y^2\right).$$

This is a cubic equation in y, and one of its solutions is $y = \frac{413}{16}$. With this choice, equation (6) becomes

$$\left(x^2 - \frac{313}{16}\right)^2 = \frac{24}{4}x^2 + \frac{595}{8}x + \frac{14161}{64} = \left(\frac{1}{8}(20x + 119)\right)^2.$$

Taking roots on both sides, we obtain the two quadratic equations

$$x^2 - \frac{313}{16} = \pm\frac{1}{8}(20x + 119).$$

For the plus sign, we find $x_1 = -\frac{19}{4}$ and $x_2 = \frac{29}{4}$, and for the minus sign, we obtain $x_3 = -\frac{15}{4}$, $x_4 = \frac{5}{4}$. This leads to $z_1 = -5$, $z_2 = 7$, $z_3 = -4$, and $z_4 = 1$.

The general case of a quartic equation. Let us now solve the general quartic equation

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + z^4 = 0,$$

by the geometrically inspired method. The coefficients are allowed to be complex numbers, and all subsequent computations are carried out in $\mathbb{C}$, i.e., we work in $\mathbb{P}^2(\mathbb{C})$ with embedded $\mathbb{C}^2$. Substituting $z = x - \mu$ yields for the quadratic term of x^2 the expression $a_2 - 3a_3\mu + 6\mu^2$. This is a quadratic equation, and we can choose one of its solutions to obtain a quartic equation without quadratic term. So, from now on, we assume that the quartic has the form

$$a_0 + a_1 x + a_3 x^3 + x^4 = 0. \tag{7}$$

Consider the two conics

$$C_1 : x^2 - y = 0, \tag{8}$$

$$C_2 : a_0 + a_1 x + a_3 xy + y^2 = 0. \tag{9}$$

As in the aforementioned example, we have that if x and y are solutions of the system (8)–(9), then x solves (7), and vice versa, if x is a solution of (7), then $x, y = x^2$ is a solution of (8)–(9). As before, the first conic C_1 is given by $\langle X, AX\rangle = 0$, and the second conic C_2 by $\langle X, BX\rangle = 0$, where

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_3 & a_1 \\ a_3 & 2 & 0 \\ a_1 & 0 & 2a_0 \end{pmatrix},$$

with $X = (x, y, 1)^\top$. The cubic equation to determine the degenerate conics in the pencil of C_1 and C_2 is

$$-\frac{1}{2}\det(\lambda A + B) = \underbrace{a_1^2 + a_0 a_3^2}_{=:p} + \underbrace{(a_1 a_3 - 4a_0)}_{=:q}\lambda + \lambda^3 = 0. \tag{10}$$

If $p = 0$, then $\lambda = 0$ is a solution. This means that B is singular and the left-hand side of (9) is the product of two linear terms, namely, $(-a_1/a_3 + a_3 x + y)(a_1/a_3 + y)$ if $a_3 \neq 0$ and $(y + \sqrt{-a_0})(y - \sqrt{-a_0})$ if $a_3 = 0$. The problem is therefore reduced to determining the intersection points of the parabola C_1 and straight lines.

Let us assume now that $p \neq 0$. Then, as the product of the solutions of (10) is $-p$, they are all nonzero. If all were equal (with nonzero value λ_0), the coefficient of λ^2 in (10) would be $-3\lambda_0 \neq 0$, whereas it is 0; thus, there are at least two different solutions. The solutions can be found by Cardano's formulas. However, these formulas are not easy to remember. We therefore follow a simple, memorizable solution method. Any nonzero $\kappa \in \mathbb{C}$ may be expressed as $\alpha + \beta$, where $\alpha^3 + \beta^3 = -p$ and $\alpha \neq 0$. Indeed, set $\beta = \kappa - \alpha$. Then, $\alpha^3 + \beta^3 = \kappa(\kappa^2 - 3\kappa\alpha + 3\alpha^2)$; one has a quadratic equation $3\alpha^2 - 3\kappa\alpha + \kappa^2 = -p/\kappa$ for α. If $\kappa^3 = -p$, the solutions are $\alpha = 0$ and $\alpha = \kappa$; take $\alpha \coloneqq \kappa$ (so that $\alpha \neq 0$) and $\beta \coloneqq 0$. If $\kappa^3 \neq -p$, then the equation for α has nonzero solutions only, and either value gives $\kappa = \alpha + \beta$ (again with $\alpha \neq 0$) as desired.

Now, write the (nonzero) solution λ of (10) as $\alpha + \beta$, where $\alpha^3 + \beta^3 = -p$ and $\alpha \neq 0$. Then, (10) becomes

$$0 = p + q(\alpha + \beta) + (\alpha + \beta)^3 = (\alpha + \beta)(q + 3\alpha\beta) = \lambda(q + 3\alpha\beta),$$

from which $q + 3\alpha\beta = 0$ (as $\lambda \neq 0$) and $\beta = -q/3\alpha$. But, therefore,

$$-p = \alpha^3 + \beta^3 = \alpha^3 - \frac{q^3}{27\alpha^3},$$

which gives finitely many possible values for α, hence for β, and so for $\alpha + \beta$; among them will be the solutions for (10). Take two distinct solutions λ_1 and λ_2.

For these values, each conic given by $\langle X, (\lambda_j A + B)X\rangle = 0$ is the union of two straight lines

$$0 = \langle X, (\lambda_j A + B)X\rangle = \langle X, g_j\rangle\langle X, h_j\rangle, \tag{11}$$

say

$$g_j = \begin{pmatrix} r_j \\ s_j \\ t_j \end{pmatrix}, \quad h_j = \begin{pmatrix} u_j \\ v_j \\ w_j \end{pmatrix}.$$

Writing out the matrix, we have

$$\lambda_j A + B = \begin{pmatrix} 2\lambda_j & a_3 & a_1 \\ a_3 & 2 & -\lambda_j \\ a_1 & -\lambda_j & 2a_0 \end{pmatrix}.$$

If we compare the coefficients in (11), we obtain

$$\begin{aligned} r_j u_j &= 2\lambda_j, & s_j u_j + r_j v_j &= 2a_3, \\ s_j v_j &= 2, & t_j u_j + r_j w_j &= 2a_1, \\ t_j w_j &= 2a_0, & t_j v_j + s_j w_j &= -2\lambda_j. \end{aligned}$$

The solution can be written as follows (plug in to check while keeping (10) in mind):

$$\begin{aligned} (u_j, v_j, w_j) &= \frac{1}{2}(2\lambda_j, a_3 + \sqrt{\mu_j}, a_1 + q_j\sqrt{v_j}), \\ (r_j, s_j, t_j) &= \frac{1}{\lambda_j}(2\lambda_j, a_3 - \sqrt{\mu_j}, a_1 - q_j\sqrt{v_j}). \end{aligned}$$

Since we are dealing with projective lines, one can drop the factors $\frac{1}{2}$ and $\frac{1}{\lambda_j}$ later. Here,

$$\mu_j := a_3^2 - 4\lambda_j \quad \text{and} \quad v_j := a_1^2 - 4a_0\lambda_j$$

are negative minors of $\lambda_j A + B$. Depending on the choice of the square roots of μ_1 and v_j, the sign q_j is

$$q_j := \frac{a_1 a_3 + 2\lambda_j^2}{\sqrt{\mu_j}\sqrt{v_j}} \in \{-1, 1\}, \quad \text{if } \mu_j v_j \neq 0\,.$$

Observe that

$$0 = -2\lambda_j \det(\lambda_j A + B) = (a_1 a_3 + 2\lambda_j^2)^2 - \mu_j v_j.$$

If $\mu_j v_j = 0$, one can choose either $q_j = 1$ or $q_j = -1$. Now that the lines g_j and h_j are determined, their intersections and hence the solutions x_j are easily computed by the respective cross-products, as in the example.

Of course, there are also other ways to find the factorization in (11).

4 The symmetric case

Let us assume that two conics C_A and C_B in the real (x_1, x_2)-plane are centrally symmetric with respect to a point $P = (p_1, p_2)$. We claim that in this case, the intersection points (real or complex) are given by quadratic equations. To see this, let C_A and C_B be given by equations $\langle X, AX\rangle = 0$ and $\langle X, BX\rangle = 0$ with matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{pmatrix}.$$

The coordinate transformation $X = T\tilde{X}$ for

$$T = \begin{pmatrix} 1 & 0 & p_1 \\ 0 & 1 & p_2 \\ 0 & 0 & 1 \end{pmatrix}$$

corresponds to a translation of the coordinates in the (x_1, x_2)-plane. In the new coordinates, the conics are centrally symmetric with respect to the origin in the $(\tilde{x}_1, \tilde{x}_2)$-plane, and they are given by $\langle \tilde{X}, \tilde{A}\tilde{X} \rangle = 0$ and $\langle \tilde{X}, \tilde{b}\tilde{X} \rangle = 0$, where

$$\tilde{A} = T^\top AT = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 \\ \tilde{a}_{12} & \tilde{a}_{22} & 0 \\ 0 & 0 & \tilde{a}_{33} \end{pmatrix} \quad \text{and} \quad \tilde{b} = T^\top BT = \begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & 0 \\ \tilde{b}_{12} & \tilde{b}_{22} & 0 \\ 0 & 0 & \tilde{b}_{33} \end{pmatrix}.$$

The condition that $\tilde{a}_{13} = \tilde{a}_{23} = 0$ yields

$$p_1 = \frac{\mathrm{cof}A_{31}}{\mathrm{cof}A_{33}} \quad \text{and} \quad p_2 = \frac{\mathrm{cof}A_{32}}{\mathrm{cof}A_{33}},$$

where $\mathrm{cof}A_{ij}$ denotes the cofactors of the matrix A, i.e.,

$$\begin{aligned} \mathrm{cof}A_{31} &= a_{12}a_{23} - a_{13}a_{22}, \\ \mathrm{cof}A_{23} &= a_{12}a_{13} - a_{11}a_{23}, \\ \mathrm{cof}A_{33} &= a_{11}a_{22} - a_{12}^2. \end{aligned}$$

In other words, the two original conics C_A and C_B are centrally symmetric, if and only if

$$\frac{\mathrm{cof}A_{31}}{\mathrm{cof}A_{33}} = \frac{\mathrm{cof}B_{31}}{\mathrm{cof}B_{33}} \quad \text{and} \quad \frac{\mathrm{cof}A_{32}}{\mathrm{cof}A_{33}} = \frac{\mathrm{cof}B_{32}}{\mathrm{cof}B_{33}}.$$

The equations $\langle \tilde{X}, \tilde{A}\tilde{X} \rangle = 0$ and $\langle \tilde{X}, \tilde{b}\tilde{X} \rangle = 0$ are

$$\tilde{a}_{11}\tilde{x}_1^2 + 2\tilde{a}_{12}\tilde{x}_1\tilde{x}_2 + \tilde{a}_{22}\tilde{x}_2^2 + \tilde{a}_{33} = 0, \tag{12}$$

$$\tilde{b}_{11}\tilde{x}_1^2 + 2\tilde{b}_{12}\tilde{x}_1\tilde{x}_2 + \tilde{b}_{22}\tilde{x}_2^2 + \tilde{b}_{33} = 0. \tag{13}$$

The linear combination $\tilde{b}_{33}$(12)–$\tilde{a}_{33}$(13) is

$$(\tilde{b}_{33}\tilde{a}_{11} - \tilde{a}_{33}\tilde{b}_{11})\tilde{x}_1^2 + 2(\tilde{b}_{33}\tilde{a}_{12} - \tilde{a}_{33}\tilde{b}_{12})\tilde{x}_1\tilde{x}_2 + (\tilde{b}_{33}\tilde{a}_{22} - \tilde{a}_{33}\tilde{b}_{22})\tilde{x}_2^2 = 0,$$

which clearly factors into two real or complex lines or a double line. Their intersection with the conic given by (12) or (13) corresponds to a quadratic problem, which yields the real or complex intersection of the two conics in $(\tilde{x}_1, \tilde{x}_2)$-coordinates, and by adding (p_1, p_2) in the original (x_1, x_2)-coordinates.

Since the intersection points in the centrally symmetric case are given by quadratic equations, it is possible to construct the intersections by ruler and compass. The reader is invited to find a construction, e.g., for the case of a circle and an ellipse. Note however, that the property “centrally symmetric” is not projectively invariant. In particular, the construction from Section 2 does not help in the symmetric case.

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