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SOLITON SOLUTIONS FOR THE MEAN CURVATURE FLOW

N. Hungerbühler and K. Smoczyk
Max-Planck Institute for Mathematics in the Sciences
Inselstrasse 22-26, 04103 Leipzig, Germany

(Submitted by: Yoshikazu Giga)

Abstract. We consider soliton solutions of the mean curvature flow, i.e., solutions which move under the mean curvature flow by a group of isometries of the ambient manifold. Several examples of solitons on manifolds are discussed. Moreover we present a local existence result for rotating solitons. We also prove global existence and stability for perturbed initial data close to a local soliton.

1. Introduction. Let N be an n-dimensional Riemannian manifold with metric $\bar{g}$. We assume that on N a Killing vector field $X : N \to TN$ exists; i.e., X generates a one-parameter group $\varphi : N \times \mathbb{R} \to N$ of isometries on N:

$$\begin{aligned} \frac{d\varphi(x,t)}{dt} &= X(\varphi(x,t)) \qquad && \text{on } N \times \mathbb{R} \\ \varphi(x,0) &= x && \text{on } N. \end{aligned}$$

Further, let $F_t : M \to N$, $t \geqslant 0$, be a smooth family of immersions of a smooth, connected m-dimensional manifold M, $m = n-1$, into N. F_t is a solution of the mean curvature flow on $(0,T)$, $T > 0$, if

$$\frac{d}{dt}F_t = -H\nu \qquad \text{on } M \times (0,T) \tag{1}$$

$$F_0 = f \qquad \text{on } M, \tag{2}$$

where $f : M \to N$ is a given initial hypersurface M_0. Here, as usual, H denotes the mean curvature of $F_t(M)$ with respect to the normal unit vector field ν on $F_t(M)$.

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Now, we say that F_t is a soliton solution of the mean curvature flow with respect to the Killing vector field X, if $\tilde{F}_t := \varphi^{-1}(F_t, t)$ is stationary in normal direction, i.e., $\tilde{F}_t(M)$ is the fixed submanifold M_0. We have

$$\begin{aligned}\frac{d}{dt}\tilde{F}_t(x) &= \frac{\partial\varphi^{-1}(F_t(x),t)}{\partial t} + D\varphi^{-1}(F_t(x),t)\frac{\partial F_t(x)}{\partial t}\\ &= -X(\tilde{F}_t(x)) - D\varphi^{-1}(F_t(x),t)H\nu.\end{aligned}$$

Because X is a Killing vector field, $\tilde{\nu}(x,t) = D\varphi^{-1}(F_t(x),t)\nu(x,t)$ is a unit normal vector field on $\tilde{F}_t(M)$. Since we assume that $\langle \frac{d}{dt}\tilde{F}_t(x), \tilde{\nu}(x,t)\rangle = 0$ we obtain $-\langle X(\tilde{F}_t), \tilde{\nu}\rangle = H\langle D\varphi^{-1}\nu, \tilde{\nu}\rangle = H\langle\tilde{\nu},\tilde{\nu}\rangle = H$. This holds in particular for $t = 0$; hence a soliton solution $f : M \to N$ of the mean curvature flow satisfies the equation

$$-\langle X(f), \nu(f)\rangle = H(f). \tag{3}$$

Rotating solitons in $\mathbb{R}^n$ will be of special interest; therefore we include the corresponding equation here: The one-parameter group $e^{At} \in SO(n,\mathbb{R})$ of rotations is generated by the Killing vector field $X(x) = Ax$, $A \in so(n,\mathbb{R})$, and hence equation (3) translates into

$$-\langle Af, \nu(f)\rangle = H(f). \tag{4}$$

For concrete calculations it is necessary to express the terms in (3) and (4) in local coordinates. We will denote the metric and the Christoffel symbols on the ambient manifold N by $\bar{g}_{\alpha\beta}$ and $\bar{\Gamma}^\alpha_{\beta\gamma}$ with Greek indices α, β, γ running from 1 to n and the metric and Christoffel symbols of the immersed manifold M by g_{ij} and Γ^k_{ij} with Latin indices i, j, k running from 1 to m. We recall the Gauss equation,

$$\frac{\partial f^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f^\alpha}{\partial x^k} + \bar{\Gamma}^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j} = -h_{ij}\nu^\alpha, \tag{5}$$

where h_{ij} is the second fundamental form. Thus, h_{ij} can be obtained by multiplying (5) by $\nu_\alpha = \bar{g}_{\alpha\beta}\nu^\beta$. Then, the mean curvature H is given by $H = g^{ij}h_{ij}$.

Group invariant properties of the curve-shortening flow in the plane are also discussed in [4]; the same problem for the affine curve-shortening flow is considered in [5].

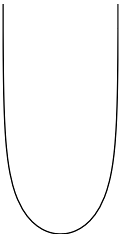

Figure 1: The grim reaper

2. Examples.

2.1. The grim reaper. Let $N = \mathbb{R}^2$ with the standard metric and $X = \binom{0}{1}$ be constant. If we parametrize the soliton solution by $f : \mathbb{R} \to \mathbb{R}^2$, $x \mapsto \binom{x}{f(x)}$, we have $\nu = \frac{1}{\sqrt{1+(f')^2}}\binom{-f'}{1}$ and $H = \frac{-f''}{(1+(f')^2)^{3/2}}$. Hence, f has to solve the equation $1 + (f')^2 = f''$. Integration yields the soliton solutions with respect to translation in direction X, $f(x) = c_1 - \log\cos(x + c_2)$ (see [1], [3] and Figure 1). Angenent [3], in the case of convex-planar curves, showed that type-II singularities are asymptotic to grim reaper curves.

2.2. Yin-yang curve. Rotating solitons in the plane have been suggested by [1]. We consider again $N = \mathbb{R}^2$ with the standard metric and equipped with Euclidean coordinates. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the generator of the rotation group. We want to parametrize the rotating soliton by $f : \mathbb{R} \to \mathbb{R}^2$, $s \mapsto \binom{x(s)}{y(s)}$, with s being the arc length. Then, the curvature is given by $H = y'x'' - x'y'' = -\langle Af', f''\rangle$ and $\nu(f) = Af'$. Hence, (4) is

$$\langle Af', f''\rangle = \langle f', f\rangle, \tag{6}$$

and multiplication by Af' yields for f the differential equation

$$f'' = Af'\langle f', f\rangle. \tag{7}$$

Denoting $u = x'$ and $v = y'$ we are left with the following first-order system:

$$\frac{d}{ds}\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ -v(xu + yv) \\ u(xu + yv) \end{pmatrix}. \tag{8}$$

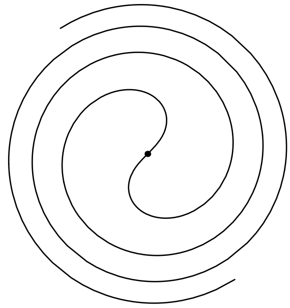

Figure 2: Yin-yang curves

In order to prove that (8) allows a global solution for arbitrary initial data $f(0) = f_0$ and $f'(0) = g_0$ with $|g_0| = 1$ we consider the equation

$$\frac{d}{ds}\begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \zeta(x, y, u, v) := \begin{pmatrix} \hat{u} \\ \hat{v} \\ -\hat{v}(x\hat{u} + y\hat{v}) \\ \hat{u}(x\hat{u} + y\hat{v}) \end{pmatrix} \tag{9}$$

with the cutoff function

$$\hat{} : \alpha \mapsto \hat{\alpha} := \begin{cases} 1 & \text{if } \alpha > 1 \\ \alpha & \text{if } -1 \leqslant \alpha \leqslant 1 \\ -1 & \text{if } \alpha < -1 \end{cases}$$

for $\alpha \in \mathbb{R}$. Then, ζ is globally Lipschitz continuous, and hence (9) admits a global solution. Since $uu' + vv' = 0$, we have $u^2 + v^2 = 1$ for all s, and hence $\hat{u} = u, \hat{v} = v$ and this is also a solution to the original equation (8). Thus, modulo rotation, we find a one-parameter family of soliton solutions which rotate with the same angular velocity (see Figure 2).

The solution has nice geometric properties, which we want to discuss now. Let us denote by

$$\beta(s) := \int_{s_0}^{s} d\beta = \int_{s_0}^{s} \langle Af', f'' \rangle ds$$

the oriented (integrated) angle of the tangent in $f(s)$ with respect to a fixed direction $f'(s_0)$ (see Figure 3).

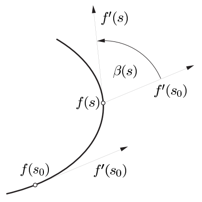

Figure 3: Geometric interpretation

Then, we observe that by (6) the quantity $\beta(s) - \frac{1}{2}f^2(s)$ has vanishing derivative. Thus we have the following result:

Lemma 1. *Along the yin-yang curve the quantity* $f^2 - 2\beta$ *is constant.*

For another geometric property of yin-yang curves compare Lemma 2 in Section 4. Using the geometric observation in Lemma 1 it is easy to prove the following properties of the rotating soliton:

Proposition 1. (a) *The yin-yang curve is embedded.* (b) *Two different yin-yang curves (rotating by the same angular velocity) intersect in at most one point.*

Proof. (a) Suppose, for some $s_1 \neq s_2$ we have $f(s_1) = f(s_2)$. Then, in particular, $f^2(s_1) = f^2(s_2)$ and Lemma 1 yields $\beta(s_1) = \beta(s_2)$. This implies that at an intersection point, the tangents agree, which is impossible since the differential equation is uniquely solvable.

(b) Suppose we have two intersection points $f_1(s_{1i}) = f_2(s_{2i})$, $i = 1, 2$, and we may assume that the intersections are transversal. By Lemma 1 we have $f_1^2(s_{11}) - \beta(s_{11}) = f_1^2(s_{12}) - \beta(s_{12})$, $f_2^2(s_{21}) - \beta(s_{21}) = f_2^2(s_{22}) - \beta(s_{22})$. Subtraction yields

$$\beta(s_{21}) - \beta(s_{11}) = \beta(s_{22}) - \beta(s_{12}). \tag{10}$$

Now, in two (consecutive) points of intersection, the right-hand side and the left-hand side of (10) have opposite sign, which is a contradiction. □

Moreover, we can make a quantitative statement about the asymptotic behavior of the solution. We consider the situation in Figure 4. From Lemma 1, we infer that $r_1^2 - 2\beta = r_2^2 - 2(\beta + \pi)$. Hence, we have

$$\Delta r := r_2 - r_1 = \frac{2\pi}{r_1 + r_2} \approx \frac{\pi}{r}.$$

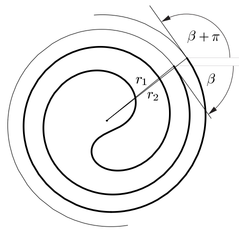

Figure 4: Asymptotic behavior

It is possible, and sometimes useful for numeric calculations, to integrate the second-order differential equation (7) once. For this purpose we consider the equation in polar coordinates and parametrize the solution by $f : \mathbb{R} \to \mathbb{R}^2$, $s \mapsto \binom{r(s)\cos(\varphi(s))}{r(s)\sin(\varphi(s))}$. Then,

$$H = -\frac{2(r')^2\varphi' + r\varphi'(r(\varphi')^2 - r'') + rr'\varphi''}{((r')^2 + r^2(\varphi')^2)^{3/2}}$$

and

$$\langle Af, \nu\rangle = \frac{rr'}{((r')^2 + r^2(\varphi')^2)^{1/2}}\,.$$

If we set $\varphi = \mathrm{id}$, equation (4) expands to

$$rr'((r')^2 + r^2) = r^2 + (r')^2 - (\frac{r'}{r})'r^2.$$

Dividing by $(r')^2 + r^2$ this is equivalent to

$$(\frac{1}{2}r^2)' = 1 - \frac{(\frac{r'}{r})'}{1 + (\frac{r'}{r})^2}\,.$$

Since the right-hand side is also a derivative, we can integrate and obtain the first-order differential equation

$$\frac{1}{2}r^2 + \arctan(\frac{r'}{r}) = s + c,$$

which again amounts to Lemma 1.

For the structure of expanding spiral-like self-similar solutions for the plane curve-shortening flow, see also [9].

2.3. Rotating solitons on surfaces of revolution. Let

$$(r,\varphi)\mapsto\begin{pmatrix}R(r)\cos(\varphi)\\ R(r)\sin(\varphi)\\ z(r)\end{pmatrix}$$

be the parametrization of a surface N which is rotationally symmetric with respect to the z-axis, $R(r)>0$. In these coordinates the metric tensor is given by

$$\bar g_{\alpha\beta}=\begin{pmatrix}(z')^2+(R')^2 & 0\\ 0 & R^2\end{pmatrix}.$$

The Christoffel symbols which we need to know in order to calculate the curvature of a curve on N are $\bar\Gamma^1_{11}=\bar g^{11}(R'R''+z'z'')$, $\bar\Gamma^1_{22}=-\bar g^{11}RR'$ and $\bar\Gamma^2_{12}=\bar\Gamma^2_{21}=R'/R$. The remaining entries are zero. We want to parametrize a soliton solution to the mean curvature flow on N by $F:s\mapsto\binom{r(s)}{\varphi(s)}$ with s being the arc length; i.e., $(r')^2((z')^2+(R')^2)+(\varphi')^2R^2=1$. Differentiation of this relation yields

$$RR'r'(\varphi')^2+(r')^3(R'R''+z'z'')+((R')^2+(z')^2)r'r''+R^2\varphi'\varphi''=0. \quad (11)$$

The unit normal at the curve F is given by

$$\nu=\frac{1}{R\sqrt{\bar g_{11}}}\begin{pmatrix}\varphi'R^2\\ -r'((z')^2+(R')^2)\end{pmatrix}.$$

Finally, the (mean) curvature of F is (see Section 1)

$$H=-\Big(\frac{d^2F^\alpha}{ds^2}\bar g_{\alpha\beta}\nu^\beta+\bar\Gamma^\alpha_{\beta\gamma}\frac{dF^\beta}{ds}\frac{dF^\gamma}{ds}\bar g_{\alpha\sigma}\nu^\sigma\Big)=\sqrt{\bar g^{11}}(2R'\varphi'+R\frac{\varphi''}{r'})$$

(where we used (11) to eliminate r''). On N the Killing vector field $X=\binom{0}{1}$ generates the isometries of N given by the rotation with respect to the symmetry axis z. Then, using (11), $H=-\langle X,\nu\rangle$ reduces to

$$\frac{d}{ds}\begin{pmatrix}r\\ \varphi\\ u\\ v\end{pmatrix}=\begin{pmatrix}u\\ v\\ \frac{v^2RR'-uvR^2((R')^2+(z')^2)-u^2(R'R''+z'z'')}{(R')^2+(z')^2}\\ u^2((R')^2+(z')^2)-2uv\frac{R'}{R}\end{pmatrix}.$$

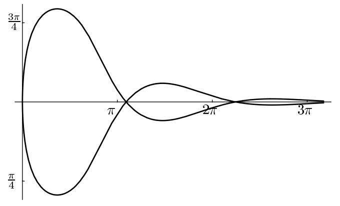

Figure 5: Solution $s \mapsto \binom{\varphi(s)}{r(s)}$

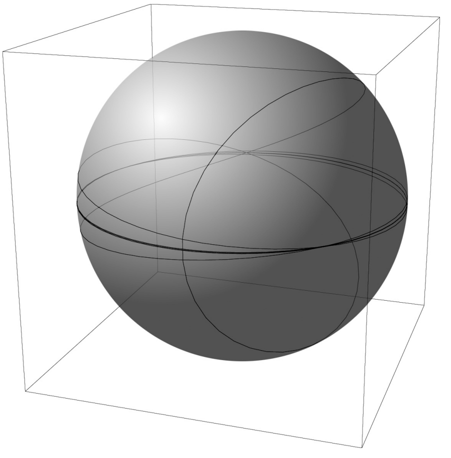

Figure 6: Rotating soliton on a sphere

For N being a sphere, i.e., $R(r) = \sin(r)$ and $z(r) = \cos(r)$, this soliton equation is

$$\frac{d}{ds}\begin{pmatrix} r \\ \varphi \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ v\sin(r)(v\cos(r) - u\sin(r)) \\ u(u - 2v\cot(r)) \end{pmatrix}.$$

Existence of a global solution is proved as for the yin-yang curve in the plane. It turns out that the solution on the sphere is not imbedded, it oscillates around the equator (see Figure 5). An example of a rotating soliton on a sphere is shown in Figure 6. On the one-sheet hyperboloid of revolution N given by $R(r) = \cosh(r)$, $z(r) = \sinh(r)$, the equation for a rotating soliton

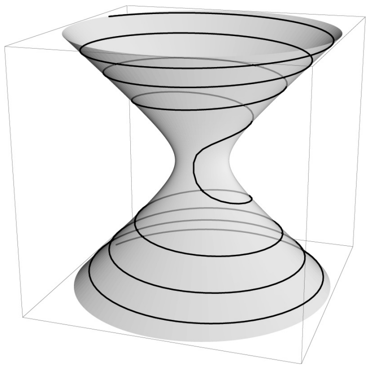

Figure 7: Rotating soliton on a hyperboloid

is

$$\frac{d}{ds}\begin{pmatrix} r \\ \varphi \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ (\frac{v^2}{2} - u^2)\tanh(2r) - uv\cosh^2(r) \\ u(u\cosh(2r) - 2v\tanh(r)) \end{pmatrix}.$$

One of the solutions is drawn in Figure 7.

2.4. Solitons on a helicoid. On the right helicoid N parametrized by $(r, \varphi) \mapsto \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ \varphi \end{pmatrix}$ the Killing vector field $X = \binom{0}{1}$ generates the isometry group of helicoidal displacement. The metric tensor is given by

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + r^2 \end{pmatrix},$$

and the nonzero Christoffel symbols are $\bar{\Gamma}^1_{22} = -r$ and $\bar{\Gamma}^2_{12} = \bar{\Gamma}^2_{21} = \frac{r}{1+r^2}$. The soliton curve is parametrized by $F : s \mapsto \binom{r(s)}{\varphi(s)}$, where s is assumed to be the arc length; i.e., $(r')^2 + (\varphi')^2(1 + r^2) = 1$. Using this, the unit-normal vector is given by $\nu = \binom{\varphi'(1+r^2)}{-r'}\frac{1}{\sqrt{1+r^2}}$ and the (mean) curvature by

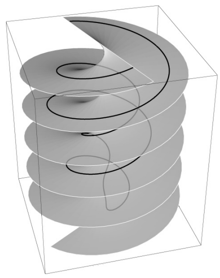

Figure 8: Screwing soliton on a helicoid

$H = -\frac{2rr'\varphi' + (1+r^2)\varphi''}{r'\sqrt{1+r^2}}$. This leads to the following equation for the soliton F:

$$\frac{d}{ds}\begin{pmatrix} r \\ \varphi \\ u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ v(vr - u(1+r^2)) \\ u^2 - 2\frac{uvr}{1+r^2} \end{pmatrix}.$$

One example of a screwing soliton is shown in Figure 8.

2.5. A two-dimensional rotating soliton. We want to construct a soliton which rotates via e^{At} in $\mathbb{R}^3$,

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $G := \{\varphi_t : t \in \mathbb{R}\}$ be the group of isometries

$$\varphi_t(p) := e^{At}p + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

acting freely and properly on $\mathbb{R}^3$ such that $\pi : \mathbb{R}^3 \to \mathbb{R}^3/G = \mathbb{R}^2$ becomes a Riemannian submersion with fibers $[p] := \{\varphi_t(p) : t \in \mathbb{R}\}$. The mean curvature vector of $[p]$ is given by $\overrightarrow{H}_{[p]} = -\frac{r}{1+r^2}e_r$, where (r, φ, z) are cylindrical coordinates for $\mathbb{R}^3$. Applying the results in [11] we obtain the evolution equation

$$\frac{d}{dt}F = -(H + \frac{r}{1+r^2}\langle e_r, \nu\rangle)\nu \tag{12}$$

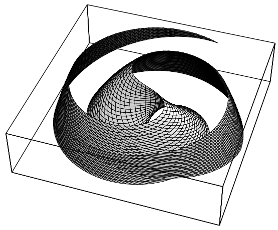

Figure 9: Two-dimensional rotating soliton

for the cross sections of a surface in $\mathbb{R}^3$ that is invariant under G and moves by its mean curvature. Proceeding similarly as for the yin-yang curve we obtain a rotating curve $\gamma \in \mathbb{R}^2$ satisfying (12). The solution is an embedded curve as is the case for the yin-yang curve. Thus the resulting surface $G\gamma \in \mathbb{R}^3$ is embedded. A slice of the rotating two-dimensional soliton can be seen in Figure 9.

3. Rotating solitons. In this section we want to prove a local existence result for rotating soliton solutions of the mean curvature flow. More precisely, we consider the following situation: The ambient manifold N is chosen to be the Euclidean space $\mathbb{R}^n$ and M is an m-dimensional hypersurface ($n = m+1$) in $\mathbb{R}^n$ which is supposed to move as a soliton solution of the mean curvature flow with respect to the Killing vector field $X(x) = Ax$, $A \in so(n, \mathbb{R})$; i.e., M satisfies equation (4). Locally, M can be represented as the graph of a real function u over a hyperplane; i.e., we parametrize M (locally) by $F : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n, x \mapsto (x, u(x))$. Using the shorthand notation $u_i := \frac{\partial u}{\partial x^i}$ we have

$$\nabla_i F = \delta_i^k \frac{\partial}{\partial x^k} + u_i \frac{\partial}{\partial u},$$

and thus we obtain $g_{ij} = g_{ij}(u) = \delta_{ij} + u_i u_j$ for the metric tensor and for its inverse

$$g^{ij} = g^{ij}(u) = \delta^{ij} - \frac{1}{w^2}\delta^{ik} u_k \delta^{jl} u_l$$

with $w^2 = w^2(u) = 1 + \delta^{ij} u_i u_j$. It is then easy to check that

$$\nu = -\frac{1}{w}(\delta^{kl} u_l \frac{\partial}{\partial x^k} - \frac{\partial}{\partial u})$$

is a unit normal on the surface. The Christoffel symbols are

$$\Gamma_{ij}^k = \nabla^k u u_{ij}$$

where we raised the index by using the metric tensor: $\nabla^k u = g^{kl}u_l$. For the mixed derivatives we obtain

$$\nabla_i\nabla_j u = u_{ij} - \Gamma_{ij}^k u_k = (1-|\nabla u|^2)u_{ij}$$

where $|\nabla u|^2 = \nabla^k u \nabla_k u$. In particular we have $w^2(1-|\nabla u|^2) = 1$. Now, by multiplying (5) by ν_α, we find

$$h_{ij} = -\frac{1}{w}u_{ij} = -w\nabla_i\nabla_j u,$$

and contraction with g^{ij} yields for the mean curvature

$$H = g^{ij}h_{ij} = -\frac{1}{w}g^{ij}u_{ij} = -w\Delta u.$$

Hence, if $\zeta \mapsto B\zeta + a$, $B \in SO(n,\mathbb{R})$, $a \in \mathbb{R}^n$, describes the coordinate transformation into the coordinate system $\binom{x}{u}$, equation (4) is in these coordinates:

$$\begin{aligned} g^{ij}u_{ij} &= -\langle BAB^{-1}(\begin{pmatrix} x \\ u \end{pmatrix} - a), \delta^{kl}u_l\frac{\partial}{\partial x^k} - \frac{\partial}{\partial u}\rangle \\ &= -\langle BAB^{-1}((x^i - a^i)\frac{\partial}{\partial x^i} + (u-a^n)\frac{\partial}{\partial u}), \delta^{kl}u_l\frac{\partial}{\partial x^k} - \frac{\partial}{\partial u}\rangle \\ &= -(x^i-a^i)\delta^{kl}u_l\langle BAB^{-1}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\rangle + (x^i-a^i)\langle BAB^{-1}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u}\rangle \\ &\quad -(u-a^n)\delta^{kl}u_l\langle BAB^{-1}\frac{\partial}{\partial u}, \frac{\partial}{\partial x^k}\rangle + (u-a^n)\langle BAB^{-1}\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\rangle. \end{aligned}$$

Using that A is anti-symmetric, we obtain

$$\begin{aligned} g^{ij}u_{ij} = -(x^i-a^i)\delta^{kl}u_l\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\rangle + \\ + (x^i - a^i + (u-a^n)\delta^{il}u_l)\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u}\rangle, \quad (13) \end{aligned}$$

where $A' = BAB^{-1}$ describes the action of A in the chosen coordinates.

Now, we want to solve equation (13) locally. For this, we choose a bounded and open set $\Omega \subset \mathbb{R}^m$ with $\partial\Omega \in C^{2,\alpha}$ and prescribe boundary data $\varphi \in C^{2,\alpha}(\bar{\Omega})$. In view of the yin-yang curve, it will be necessary to implement a smallness parameter. We do this by replacing A by ωA and choosing the angular velocity $\omega \in \mathbb{R}$ sufficiently small later. So, the equation we want to solve in Ω (for $|\omega|$ small enough) is

$$g^{ij}u_{ij} = -(x^i - a^i)\delta^{kl}u_l\omega\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\rangle + \\ + (x^i - a^i + (u - a^n)\delta^{il}u_l)\omega\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u}\rangle, \quad (14)$$

with $A' = BAB^{-1}$ and with $u = \varphi$ on $\partial\Omega$.

We prove existence of a solution of the Dirichlet problem (14) by Banach's fixed-point theorem: For $k \in (0, 1)$ (to be chosen later) and fixed $\alpha \in (0, 1)$, we consider the set $X := \{f \in C^{2,\alpha}(\bar{\Omega}) : \|f\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant k, f = \varphi \text{ on } \partial\Omega\}$. This set equipped with the $C^{2,\alpha}$-topology is a complete metric space. For $u \in X$ we define $T(u)$ to be the unique solution $v \in C^{2,\alpha}(\bar{\Omega})$ of the linear Dirichlet problem

$$g^{ij}(u)v_{ij} = -(x^i - a^i)\delta^{kl}u_l\omega\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\rangle + \\ + (x^i - a^i + (u - a^n)\delta^{il}u_l)\omega\langle A'\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u}\rangle \qquad \text{in } \Omega \qquad (15)$$

$$u = \varphi \qquad \text{on } \partial\Omega. \qquad (16)$$

Observe that the right-hand side of (15) belongs to $C^{1,\alpha}(\bar{\Omega})$ for $u \in X$ and that $g^{ij}(u)$ defines a uniformly elliptic operator on Ω for $k \in (0, 1)$. In fact, we have

$$\begin{aligned} g^{ij}(u)\xi_i\xi_j &= |\xi|^2 - \frac{1}{1 + |\nabla u|^2}|\nabla u \cdot \xi|^2 \\ &= \frac{1}{1 + |\nabla u|^2}(|\xi|^2 + |\nabla u|^2|\xi|^2 - |\nabla u \cdot \xi|^2) \geqslant \frac{1}{2}|\xi|^2, \end{aligned}$$

and of course we have $\|g^{ij}(u)\|_{C^{\alpha}(\bar{\Omega})} \leqslant C$ for a constant C which can be chosen independent of $k \in (0, 1)$. Thus, by Kellogg's theorem (see, e.g., [8]) it follows that $v = T(u)$ is well defined.

Now, we show that T maps X into X provided the quantity $|\omega| + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})}$ is sufficiently small. To this end we observe that the right-hand side of (15), which we want to denote by $f(u)$, satisfies $\|f(u)\|_{C^{\alpha}(\bar{\Omega})} \leqslant C|\omega|$ for a constant C which depends only a, A, B and Ω but not on $k \in (0, 1)$.

Then, again by standard elliptic theory (see, e.g., [8, Theorems 6.6 and 6.8]), $v = T(u)$ satisfies for $u \in X$ the Hölder estimate

$$\|v\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C'(\|f(u)\|_{C^{\alpha}(\bar{\Omega})} + \|v\|_{L^{\infty}(\Omega)} + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})}) \leqslant C''(|\omega| + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})})$$

with C' and C'' independent of $k \in (0,1)$. Hence, we conclude that $v \in X$ if we choose

$$|\omega| + \|\varphi\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant \frac{k}{C''}. \tag{17}$$

It remains to prove that $T : X \to X$ is contracting: Subtraction of the two equations for $\bar{u} := T(u)$ and $\bar{v} := T(v)$ yields

$$\begin{aligned} g^{ij}(u)\bar{u}_{ij} - g^{ij}(v)\bar{v}_{ij} = f(u) - f(v) =: h \qquad & \text{in } \Omega \\ \bar{u} - \bar{v} = 0 \qquad & \text{on } \partial\Omega. \end{aligned}$$

This can be rewritten as

$$g^{ij}(u)(\bar{u}_{ij} - \bar{v}_{ij}) = h - (g^{ij}(u) - g^{ij}(v))\bar{v}_{ij} \qquad \text{in } \Omega \tag{18}$$

$$\bar{u} - \bar{v} = 0 \qquad \text{on } \partial\Omega. \tag{19}$$

It is easy to see that for arbitrary $u, v \in X$

$$\|g^{ij}(u) - g^{ij}(v)\|_{C^{\alpha}(\bar{\Omega})} \leqslant C\|u - v\|_{C^{2,\alpha}(\bar{\Omega})}.$$

A further elementary estimation shows that

$$\|h\|_{C^{\alpha}(\bar{\Omega})} = \|f(u) - f(v)\|_{C^{\alpha}(\bar{\Omega})} \leqslant C|\omega|\, \|u - v\|_{C^{2,\alpha}(\bar{\Omega})}$$

for a constant C depending only on a, A, B, Ω (but not on $k \in (0,1)$). Thus, the C^{α}-norm of the right-hand side of (18) can be estimated by $(C|\omega| + Ck)\|u - v\|_{C^{2,\alpha}(\bar{\Omega})}$. Hence, as above, elliptic theory for (18)–(19) yields

$$\|\bar{u} - \bar{v}\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C'''(|\omega| + k)\|u - v\|_{C^{2,\alpha}(\bar{\Omega})}.$$

Thus, $T : X \to X$ is a contraction if

$$|\omega| + k < \frac{1}{C'''}. \tag{20}$$

Thus, we may start by setting $k = \frac{1}{2C'''}$ and then choosing $|\omega|$ and $\|\varphi\|_{C^{2,\alpha}(\bar{\Omega})}$ so small that both (17) and (20) are satisfied. Then, $T : X \to X$ has a fixed point in X that is a solution of (14) on Ω with boundary data φ on $\partial\Omega$. By differentiating (14) we see that $u \in C^{\infty}(\Omega)$. Thus, we have proved the following theorem:

Theorem 1. *Let $\Omega \subset \mathbb{R}^m$ be bounded and open with $\partial\Omega \in C^{2,\alpha}$. Then, with the same notation as above, there exists a constant $\varepsilon > 0$, depending only on Ω, A, B and a such that the following is true: If $\varphi \in C^{2,\alpha}(\bar{\Omega})$ with $\|\varphi\|_{C^{2,\alpha}(\bar{\Omega})} < \varepsilon$ and if $|\omega| < \varepsilon$ then the soliton equation* (14) *has a solution $u \in C^{2,\alpha}(\bar{\Omega}) \cap C^{\infty}(\Omega)$ on Ω with boundary data $u = \varphi$ on $\partial\Omega$.*

We remark that a somewhat shorter proof is possible by using the Schauder fixed-point theorem. However, we prefer the above-given proof since it indicates the strategy to solve the problem numerically (see below).

We do not want to enter a detailed discussion about uniqueness here, but we should not miss drawing the following conclusion, which follows immediately from Proposition 1:

Corollary 1. *For $m = 1$ the solution in Theorem* 1 *is unique.*

We should also note that if and only if the eigenvalues of e^{tA} are commensurable (which is automatic for $m \leqslant 2$), then the solution in Theorem 1 is periodic in time.

As an example, we consider a solution of the mean curvature flow which rotates in a torus with given rotating boundary data: We take

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

which corresponds to rotation around the y-axis. Ω is chosen to be the unit circle K with center $(\frac{4}{3}, 0)$ in the (x, y)-plane; i.e., $B =$ identity and $a = 0$. For simplicity, we consider zero boundary data, $\varphi \equiv 0$, and $\omega = 1$. In other words, we look for a surface which moves under the mean curvature flow by rotation around the y-axis with angular velocity $\omega = 1$ and with rotating boundary given by the circle K which generates the torus. For the sake of completeness, we include the corresponding explicit form of the nonlinear equation (14) which describes the solution:

$$(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} = \omega(x + uu_x)(1 + u_x^2 + u_y^2).$$

Numerically, this equation can be treated by implementing the iteration

$$u^{(0)} = \varphi \tag{21}$$

$$u^{(k+1)} = T(u^{(k)}). \tag{22}$$

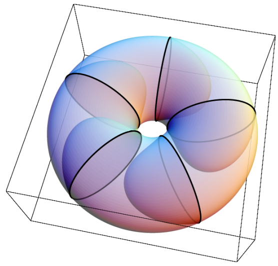

Figure 10: Rotating soliton in a torus

The solution $u^{(k+1)}$ of the elliptic equation (22) may be approximated by performing one Gauss-Seidel step. Then iteration together with a successive refinement of the grid yields a reasonably fast-converging algorithm. A picture of the solution in five positions is drawn in Figure 10.

4. Global existence for rotating boundary data. In the previous section, we proved existence of rotating solitons with given rotating boundary data. In this section we use a perturbation of such a solution as initial data and show that the resulting mean curvature flow (still with the same rotating boundary data) admits a global solution in time. We restrict ourselves to the case $N = \mathbb{R}^2$.

It is not a priori clear what coordinate system is best suited for our purpose. We will choose a coordinate system which rotates synchronously with a yin-yang curve. In this rotating system, equation (1) has the form

$$\frac{d}{dt}F_t = -H\nu - AF \tag{23}$$

such that the yin-yang curves (associated with the chosen generator A as in Section 2.2) are just the stationary solutions of this equation.

Now, we try to parametrize the solution of the mean curvature flow equation (23) as a graph over a yin-yang curve (comparable to [7], where the solution of the mean curvature flow is considered as the graph over a plane). So, let $f : \mathbb{R} \to \mathbb{R}^2$ denote a curve (e.g. a yin-yang curve) in the plane which is parametrized by its arc length. Then, in a tubular neighborhood of the curve, we consider the coordinates $\varphi : (u, v) \mapsto f(u) + vAf'(u)$ (see Figure 11). Then we have $\varphi_u(u, v) = f'(u) + vAf''(u) = f'(u)(1 + \kappa(u)v)$ (since $\nu = Af'$ and $f'' = -\kappa\nu$ with κ the curvature of f) and $\varphi_v(u, v) = Af'(u)$, and hence

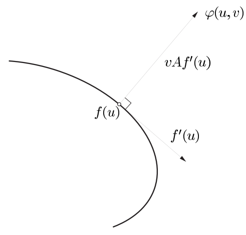

Figure 11: Rotating coordinate system

the metric tensor in the (u,v)-plane is given by

$$\bar{g}_{\alpha\beta}(u,v) = \begin{pmatrix} \bar{g} & 0 \\ 0 & 1 \end{pmatrix}$$

with $\bar{g} = \bar{g}(u,v) = \langle \varphi_u, \varphi_u \rangle = (1+\kappa(u)v)^2$. The Christoffel symbols are $\bar{\Gamma}^1_{11} = \frac{\bar{g}_u}{2\bar{g}}$, $\bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = \frac{\bar{g}_v}{2\bar{g}}$ and $\bar{\Gamma}^2_{11} = -\frac{1}{2}\bar{g}_v$ (the remaining Christoffel symbols vanish), where the indices denote partial derivatives with respect to the indexed variable.

From now on, we have to deal with the (modified) mean curvature flow (23) in the (u,v)-plane equipped with the metric $\bar{g}_{\alpha\beta}$. The mean curvature flow is thus described by a family F_t of smooth curves in the (u,v)-plane:

$$F_t : x \mapsto \begin{pmatrix} u(x,t) \\ v(u(x,t),t) \end{pmatrix}.$$

The normal vector to the curve $F_t(\cdot)$ is $\nu = \frac{1}{\sqrt{\bar{g}(v_u^2+\bar{g})}}\binom{-v_u}{\bar{g}}$, where $\bar{g}$ is evaluated at the point $F_t(u)$. For the (mean) curvature of the curve $F_t(\cdot)$ we obtain by using (5) the expression

$$H = \frac{\bar{g}\big(\bar{g}(\bar{g}_v - 2v_{uu}) + v_u(\bar{g}_u + 2\bar{g}_v v_u)\big)}{2\big(\bar{g}(\bar{g}+v_u^2)\big)^{3/2}}.$$

The total time derivative of F_t is given by

$$\frac{d}{dt}F_t = \begin{pmatrix} u_t \\ v_u u_t + v_t \end{pmatrix}.$$

In order to finish the translation of (23) into the (u,v)-coordinates it remains to find the representation of the tangent vector AF: Let $\gamma : s \mapsto$

$\varphi(u(s), v(s)) = f(u(s)) + v(s)Af'(u(s))$ be a curve with tangent vector

$$\frac{d}{ds}\gamma = f'(u)u' + v'Af'(u) + vu'Af''(u) \overset{!}{=} A\gamma.$$

Multiplication of this last relation by $f'(u)$ yields

$$u' + vu'\langle Af''(u), f'(u)\rangle = \langle Af(u), f'(u)\rangle - v, \tag{24}$$

and multiplication with $f''(u)$ gives

$$v'\langle Af'(u), f''(u)\rangle = \langle Af(u), f''(u)\rangle. \tag{25}$$

Solving (24) and (25) for u' and v' gives for the tangent vector AF the (u, v)-coordinates

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{\langle Af(u), f'(u)\rangle - v}{1 + v\langle Af''(u), f'(u)\rangle} \\ \frac{\langle Af(u), f''(u)\rangle}{\langle Af'(u), f''(u)\rangle} \end{pmatrix},$$

where we have to set $u = u(x, t)$ and $v = v(u(x, t), t)$. Hence, equation (23) is transformed into

$$\begin{aligned} u_t &= -H\nu^1 - \frac{\langle Af(u), f'(u)\rangle - v}{1 + v\kappa} \\ v_t &= -H\nu^2 - \langle f(u), f'(u)\rangle - v_u u_t. \end{aligned}$$

Eliminating u_t from these equations, we find the following equation for $v(u, t)$:

$$v_t = \frac{v_{uu}}{\bar{g} + v_u^2} - \frac{\bar{g}\bar{g}_v + v_u(\bar{g}_u + 2\bar{g}_v v_u)}{2\bar{g}(\bar{g} + v_u^2)} - \langle f, f'\rangle + v_u \frac{\langle Af, f'\rangle - v}{1 + v\kappa} \tag{26}$$

where $\bar{g} = \bar{g}(u, v(u, t))$ and $f = f(u)$, etc. In particular, if f is a yin-yang curve, then $v \equiv 0$ should be a solution of (26). In fact, for $v \equiv 0$ we have $\bar{g} = 1$, $\bar{g}_v = 2\langle Af'', f'\rangle = 2\kappa$, and (26) becomes

$$\langle Af, f''\rangle = \langle Af', f''\rangle^2 \tag{27}$$

which is easily seen to be equivalent to equation (7) which describes yin-yang curves $f'' = Af'\langle f, f'\rangle$. So, as a byproduct, we obtain from (27) another geometric interpretation of the equation of the yin-yang curves, namely:

Lemma 2. *The yin-yang curves satisfy the equation*

$$\langle Af, f'' \rangle = \kappa^2,$$

where κ denotes the curvature.

If f is a yin-yang curve, equation (26) simplifies a bit in view of (27) and is then

$$v_t = \frac{v_{uu}}{\bar{g} + v_u^2} - \frac{\bar{g}_v}{2\bar{g}} - \frac{v_u(\bar{g}_u + \bar{g}_v v_u)}{2\bar{g}(\bar{g} + v_u^2)} + \kappa + v_u \frac{\langle Af, f' \rangle - v}{1 + v\kappa}. \tag{28}$$

This is a nonlinear parabolic equation, and we now want to prove global existence in time for the initial- and boundary-value problems. Let $f : \mathbb{R} \to \mathbb{R}^2$ be a yin-yang curve parametrized by arc length, and let $P = f(p)$ and $Q = f(q)$ be two points on the curve. Furthermore, let $f_P^\pm$ and $f_Q^\pm$ be four more yin-yang curves such that

- f, f_P^+, f_P^- intersect in P.
- f, f_Q^+, f_Q^- intersect in Q.
- $f_P^\pm$ and $f_Q^\pm$ can be represented as graphs over f on the interval $[p, q]$; i.e., there exist functions $\xi_P^\pm$ and $\xi_Q^\pm$ such that $u \mapsto \varphi(u, \xi_P^\pm(u))$ is a reparametrization of $f_P^\pm$ and $u \mapsto \varphi(u, \xi_Q^\pm(u))$ is a reparametrization of $f_Q^\pm$ on the interval $u \in [p, q]$.
- $\xi_P^+ \geqslant 0$, $\xi_Q^+ \geqslant 0$, $\xi_P^- \leqslant 0$, $\xi_Q^- \leqslant 0$ on $[p, q]$.

Such curves exist by Proposition 1 and by continuous dependence of the solution of ordinary differential equations from initial data.

As we have seen in (28), the equation of the mean curvature flow in the rotating coordinate system over a yin-yang curve is of the form

$$v_t = \frac{v_{uu}}{\bar{g} + v_u^2} + \Phi(u, v, v_u), \tag{29}$$

and $v(u, t) = 0$, $v(u, t) = \xi_P^\pm(u)$ and $v(u, t) = \xi_Q^\pm(u)$ are stationary solutions of (29). Now, we try to find a global solution in time of (29) with initial condition

$$v(u, 0) = v_0 \qquad \text{for } u \in [p, q] \tag{30}$$

and with boundary condition

$$v(p,t) = v(q,t) = 0 \qquad \text{for } t \geqslant 0. \tag{31}$$

We assume that v_0 is a smooth function on $[p,q]$ which satisfies

- $v_0(p) = v_0(q) = 0$
- $g := \max\{\xi_Q^-, \xi_P^-\} \leqslant v_0 \leqslant \min\{\xi_P^+, \xi_Q^+\} =: h.$

Now, we consider the interval

$$I = \{t \geqslant 0 : \text{there exists a smooth solution of (29)–(31) on } [0,t]\}.$$

By local solvability of (29) we have $I \neq \varnothing$. By the maximum principle the short-time solution $v(u,t)$ is bounded by g from below and by h from above. In particular, $|v_u(p,\cdot)|$ and $|v_u(q,\cdot)|$ are uniformly bounded. In order to find a uniform bound for $|v_u(u,\cdot)|$ on (p,q) we proceed as follows: We consider the set $S := \{(u,v) : p \leqslant u \leqslant q, g(u) \leqslant v \leqslant h(u)\}$ and the set T of stationary solutions $\xi(u)$ which satisfy for $u \in (p,q)$: if $\xi(a) = 0$ then $\xi_u(a) = C$. Here, $C > 0$ is chosen small enough to ensure that all $\xi \in T$ are graphs over $[p,q]$ with $0 < C_1 < \xi'(u) < C_2 < \infty$ whenever $(u, \xi(u)) \in S$. If we additionally assume that $v_0' \leqslant C_1$ on $[p,q]$ it follows that v_0 intersects all $\xi \in T$ in S at exactly one point. Now, we claim that C_2 is a uniform upper bound for v_u on $]p,q[$: In fact, suppose for the sake of contradiction that we find a point $u_0 \in (p,q)$ and a time $t_0 > 0$ with $v_u(u_0,t_0) > C_2$; then $v(\cdot,t_0)$ has at least three intersection points with the stationary solution $\xi \in T$ through the point $(u_0, v(u_0,t_0))$. This contradicts the fact that the number of intersection points decreases in time (see [2]). The same arguments allow us to construct a lower bound for v_u. We conclude that v_u is uniformly bounded on $[p,q] \times I$. By the linear theory of uniformly parabolic equations (see [10, Theorem IV.9.1 and Lemma II.3.3]) and the usual Schauder iteration argument, it follows that v extends smoothly to $\bar{I}$. Thus $I \subset \mathbb{R}_+$ is open, closed and nonempty, which implies that $I = \mathbb{R}_+$.

We summarize the obtained result in the following theorem:

Theorem 2. *If P and Q are two points on a yin-yang curve γ such that the arc connecting the two points is short enough, then the following is true: There exists an $\varepsilon > 0$ such that for each initial curve connecting P and Q which has C^1 distance to γ smaller than ε, the mean curvature flow with rotating boundary has a global solution in time.*

For an existence result for properly embedded initial curves which divide the plane into two regions of infinite area, see [6].

5. Stability results for rotating solitons. In this section, we show that the solution of the mean curvature flow, which we constructed in the previous section, converges to a rotating solution for $t \to \infty$. We call this property *local stability.* We continue to consider the equation of the mean curvature flow in rotating coordinates, i.e., in the form (29). What we want to prove is that $v(\cdot, t) \to 0$ in $L^\infty([p, q])$ for $t \to \infty$. By the parabolic maximum principle, it is enough to show this for a sub- and a supersolution. We show it for a supersolution; the arguments for the subsolution are the same.

The idea is to replace v_0 by its upper envelope with respect to the family of stationary solutions. More precisely, we define

$$\hat{v}_0(u) := \inf\{\xi(u) \,:\, \xi \geqslant v_0 \text{ on } [p, q],\, \xi \text{ is a stationary solution of (29)}\}.$$

Note that, since $\xi_P^+ \geqslant v_0$ on $[p, q]$, $\hat{v}_0$ is well defined. Also note that the set $S_0 := \{(u, v) \,:\, p \leqslant u \leqslant q, 0 \leqslant v \leqslant \hat{v}_0(u)\}$ is convex with respect to the set of stationary solutions; i.e., every stationary solution through an inner point of S_0 intersects the boundary ∂S_0 in exactly two points.

We denote the solution of the mean curvature flow with initial data $\hat{v}_0$ by $\hat{v}(u, t)$. For $a \in [p, q]$, let $\xi^{a,t}$ denote the stationary solution which is defined by

$$\xi^{a,t}(a) = \hat{v}(a, t) \tag{32}$$

$$\xi_u^{a,t}(a) = \hat{v}_u(a, t). \tag{33}$$

By definition of $\hat{v}_0$ we have

$$(\hat{v}_0)_{uu}(a) \leqslant \xi_{uu}^{a,0}(a) \tag{34}$$

for all $a \in [p, q]$. We claim that the relation (34) holds for all time; i.e.,

$$\hat{v}_{uu}(a, t) \leqslant \xi_{uu}^{a,t}(a) \tag{35}$$

In fact, suppose for the sake of contradiction that (35) fails to be true for some positive time $t_0 > 0$ at a point a_0. This implies that the set $S_{t_0} := \{(u, v) \,:\, p \leqslant u \leqslant q, 0 \leqslant v \leqslant \hat{v}(u, t_0)\}$ is not convex with respect to the set

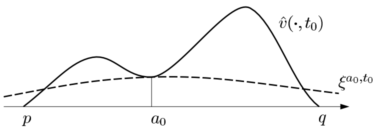

Figure 12: Nonconvex set S_{t_0}

of stationary solutions (see Figure 12). Hence, ξ^{a_0,t_0} intersects $\hat{v}(\cdot, t_0)$ at at least three points. This contradicts the fact that the number of intersections decreases in time (see [2]).

From (35) it follows that

$$\begin{aligned} \hat{v}_t(a,t) &= \frac{\hat{v}_{uu}(a,t)}{g + \hat{v}_u^2(a,t)} + \Phi(a, \hat{v}(a,t), \hat{v}_u(a,t)) \\ &\leqslant \frac{\xi_{uu}^{a,t}(a)}{g + (\xi_u^{a,t}(a))^2} + \Phi(a, \xi^{a,t}(a), \xi_u^{a,t}(a)) = 0. \end{aligned}$$

Hence, $0 \leqslant \hat{v}$ is monotone decreasing in t and must hence converge to some function $V(u) \geqslant 0$ on $[p, q]$. By the uniform estimates obtained in the previous section we can find a sequence $t_n \to \infty$ such that

$$\hat{v}_t(\cdot, t_n) \to 0 \quad \text{in } L^1([p,q]), \qquad \hat{v}(\cdot, t_n) \to V \quad \text{in } C^{2,\alpha}([p,q]).$$

Passing to the limit $t_n \to \infty$ in the equation it follows that V is a stationary solution, and by the uniqueness result stated in Corollary 1, it follows that $V \equiv 0$. This completes the proof of the following theorem:

Theorem 3. *The rotating solitons of the mean curvature flow are locally stable.*

6. A growth estimate. Let $F : \mathbb{R} \to \mathbb{R}^2$ be a smooth immersion of a curve $\gamma = F(\mathbb{R})$, and let $s \in \mathbb{R}$ denote the curve parameter. The derivatives with respect to s will be abbreviated by a prime. We set $g := |F'|^2$, $e := \frac{1}{\sqrt{g}} F'$, $\nu := -J(e)$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the complex structure on $\mathbb{R}^2$. For a fixed vector $W \in \mathbb{R}^2$ with $|W| = 1$ we have a uniquely defined smooth function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\cos\alpha = \langle W, \nu \rangle$, $\sin\alpha = -\langle W, e \rangle$, $\alpha(0) \in [0, 2\pi)$.

The Weingarten equation gives $\nu' = HF' = H\sqrt{g}e$. Then

$$\begin{aligned} d\cos\alpha &= -\sin\alpha\, d\alpha = (\cos\alpha)'\, ds = \langle W, \nu'\rangle\, ds = H\sqrt{g}\langle W, e\rangle\, ds \\ &= -H\sqrt{g}\sin\alpha\, ds = -\sin\alpha h, \end{aligned}$$

where h denotes the curvature 1-form on γ. Thus

$$d\alpha = h. \tag{36}$$

We recall some of the well-known evolution equations for the curve shortening flow in $\mathbb{R}^2$.

$$\frac{d}{dt}F = \Delta F \tag{37}$$

$$\frac{d}{dt}g = -2H^2 g \tag{38}$$

$$\frac{d}{dt}H = \Delta H + H^3 \tag{39}$$

$$\frac{d}{dt}\nu = \nabla H = \Delta\nu + H^2\nu. \tag{40}$$

This implies

$$\frac{d}{dt}\cos\alpha = -\sin\alpha\frac{d}{dt}\alpha = \langle\frac{d}{dt}\nu, W\rangle = \langle\nabla H, W\rangle = \frac{H'}{\sqrt{g}}\langle e, W\rangle = -\Delta\alpha\sin\alpha.$$

Hence

$$\frac{d}{dt}\alpha = \Delta\alpha. \tag{41}$$

We also mention that

$$\frac{d}{dt}(|F|^2 + 2t) = \Delta(|F|^2 + 2t). \tag{42}$$

An interesting function u is given by $u := \alpha - \frac{|F|^2}{2} - t$. By equations (41) and (42) u satisfies

$$\frac{d}{dt}u = \Delta u. \tag{43}$$

For the yin-yang curve we compute

$$du = d\alpha - \langle F, F'\rangle\, ds = (H - \langle F, e\rangle)\sqrt{g}\, ds = (H + \langle JF, \nu\rangle)\sqrt{g}ds = 0.$$

Therefore, $u = \text{const} \iff F$ is a yin-yang curve.

Theorem 4. *If for some $c_0 < \infty$, $p \geqslant 0$, the inequality $u^2 \leqslant c_0(1+|F|^2)^p$ is satisfied on γ_0, then for all $t > 0$, $u^2 \leqslant c_0(1+|F|^2+(2+4p)t)^p$.*

Proof. Let us set $\eta := 1+|F|^2+(2+4p)t$. Using (42) and (43) we obtain

$$\begin{aligned}\frac{d}{dt}(u^2\eta^{-p}) &= \Delta(u^2\eta^{-p}) - p(p+1)u^2\eta^{-p-2}|\nabla\eta|^2 + 4pu\eta^{-p-1}\langle\nabla u, \nabla\eta\rangle \\ &- 2\eta^{-p}|\nabla u|^2 - 4p^2u^2\eta^{-p-1}.\end{aligned}$$

Young's inequality gives

$$4pu\eta^{-p-1}\langle\nabla u, \nabla\eta\rangle \leqslant 2\eta^{-p}|\nabla u|^2 + 2p^2u^2\eta^{-p-2}|\nabla\eta|^2.$$

But $F = \langle F, e\rangle e + \langle F, \nu\rangle\nu = \frac{1}{2}\nabla|F|^2 + \langle F, \nu\rangle\nu$ which implies that

$$\eta \geqslant |F|^2 = \frac{1}{4}|\nabla|F|^2|^2 + \langle F, \nu\rangle^2 \geqslant \frac{1}{4}|\nabla|F|^2|^2 = \frac{1}{4}|\nabla\eta|^2.$$

With these inequalities we deduce

$$(\frac{d}{dt} - \Delta)(u^2\eta^{-p}) \leqslant 0,$$

and the proposition follows from Corollary 1.1 in [7].

We end this paper by computing the evolution equation for $\langle X, \nu\rangle = \langle JF, \nu\rangle$. With equations (37) and (40) we conclude

$$\begin{aligned}\frac{d}{dt}\langle JF, \nu\rangle &= \langle\Delta(JF), \nu\rangle + \langle JF, \Delta\nu + H^2\nu\rangle \\ &= \Delta\langle JF, \nu\rangle - 2\frac{1}{g}\langle JF', \nu'\rangle + H^2\langle JF, \nu\rangle,\end{aligned}$$

and since $\langle JF', \nu'\rangle = H\langle JF', F'\rangle = 0$ we obtain

$$\frac{d}{dt}\langle JF, \nu\rangle = \Delta\langle JF, \nu\rangle + H^2\langle JF, \nu\rangle. \tag{44}$$

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