Received: 11 February 2025 | Revised: 25 August 2025 | Accepted: 10 September 2025

DOI: 10.1002/zamm.70225

ZAMM

ORIGINAL PAPER

Poncelet curves

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We examine pairs of closed plane curves that have the same closing property as two conic sections in Poncelet's porism. We show how the vertex curve can be computed for a given envelope and vice versa. Our formulas are universal in the sense that they produce all possible sufficiently regular pairs of such Poncelet curves. We arrive at similar results for sets of curves, analogous to the pencil of conic sections in the full Poncelet theorem. We also study the case of Poncelet curves that carry Poncelet polygons which are equiangular or even congruent.

1 | INTRODUCTION

A popular version of Poncelet's porism reads as follows:

Theorem 1. *Let K and C be nondegenerate conics in general position which neither meet nor intersect. Suppose there is an n-sided polygon inscribed in K and circumscribed about C. Then for any point P of K, there exists an n-sided polygon, also inscribed in K and circumscribed about C, which has P as one of its vertices.*

See, for example, Halbeisen and Hungerbühler [1] for an elementary proof of the theorem and refs. [2, 3] as general references. In this article, we investigate pairs of curves which have the same closing property as the conics K and C in Theorem 1:

Definition 2. Let K and C be closed curves in the Euclidean plane $\mathbb{R}^2$. If every point $Q \in K$ is a vertex of an n-gon P inscribed in K and circumscribed about C, then (K, C) is called a *Poncelet pair*, and P a *Poncelet polygon*. The curve K is called *vertex curve*, the curve C is called *envelope*. Here, circumscribed also allows that the prolongation of the sides of P are tangential to C.

There is also the following, lesser known, full form of Poncelet's theorem (see ref. [2, Theorem 5.2] and Figure 1 on the left):

Theorem 3. *Let the conics $K, C_1, \dots, C_n$ belong to a pencil. If a polygon P inscribed in K exists such that each of its sides is tangent to one of the conics $C_1, \dots, C_n$, then infinitely many such polygons exist which touch the conics $C_1, \dots, C_n$ in the same order as P.*

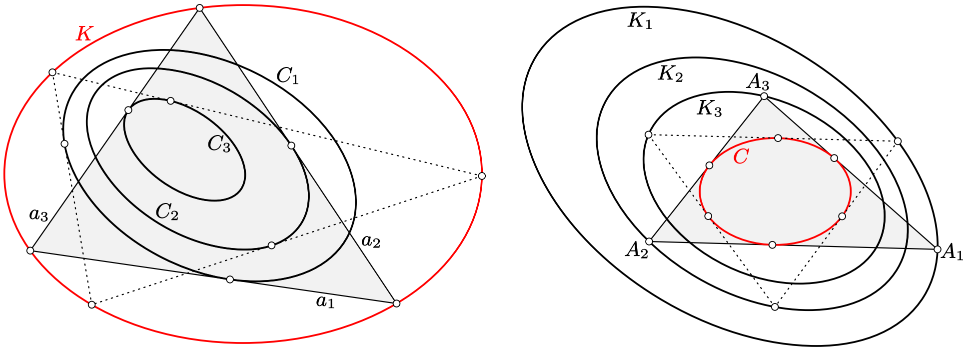

FIGURE 1 Theorem 3 for triangles with sides a_1, a_2, a_3 tangent to C_1, C_2, C_3 on the left, and Theorem 4 for triangles with vertices A_1, A_2, A_3 on K_1, K_2, K_3 on the right.

The dual form of this theorem is usually not mentioned in the literature. The reason being that the dual of the pencil generated by two conics is *not* the pencil generated by the duals of the two conics (see ref. [4]). Therefore the dual of Theorem 3 is slightly less elegant compared to its original (see Figure 1 on the right):

Theorem 4. *Let $C, K_1, \dots, K_n$ be conics such that their duals belong to a pencil. If a polygon P circumscribed about C exists such that each of its vertices belongs to one of the conics $K_1, \dots, K_n$, then infinitely many such polygons exist which have vertices on the conics $K_1, \dots, K_n$ in the same order as P.*

Definition 5. Generalizing Definition 2 we will call curves $(K, C_1, \dots, C_n)$ which have the closing property described in Theorem 3 or curves $(C, K_1, \dots, K_n)$ which have the closing property in Theorem 4 a *Poncelet clan.*

For the case when the vertex curve K is a circle, Poncelet pairs (K, C) were first considered in ref. [5] and ref. [6] in the context of numerical ranges of matrices, and later intensively studied in refs. [7–11] and [12, §5 and §7] and the survey paper [13]. Another approach to Poncelet pairs (K, C) where K is again a circle and C an algebraic curve uses Szegő polynomials and is described in ref. [14]. For a circle K, the corresponding envelope C may also be constructed using Blaschke products (see refs. [15]). While these methods seem to be limited to the special case where the vertex curve K is a circle, we will take a more direct geometric point of view here and consider methods to generate general Poncelet pairs and clans.

This article is organized as follows. In Section 2, we consider Poncelet pairs and clans with particularly nice geometric properties. In Section 3, we will construct Poncelet pairs and clans when either the vertex curve is prescribed, or when the envelope is prescribed. The formulas we develop are universal in the sense that any sufficiently regular Poncelet pair can be generated using the approach we present.

2 | EQUIANGULAR PONCELET POLYGONS

In Section 2.1, we will consider Poncelet pairs (K, C) with Poncelet polygons whose external angles are all identical. Theorem 6 describes the general solution for this situation. A special case occurs when the Poncelet polygons of (K, C) are not only equiangular, but even equilateral and all congruent. This situation is covered in Theorem 9. In Section 2.2, we will consider equiangular Poncelet polygons for Poncelet clans $(C, K_1, K_2, \dots, K_n)$.

2.1 | Equiangular pairs

If the external angles of each Poncelet polygon are equal and independent of the starting point, then we say that the Poncelet pair (K, C) is *equiangular*.

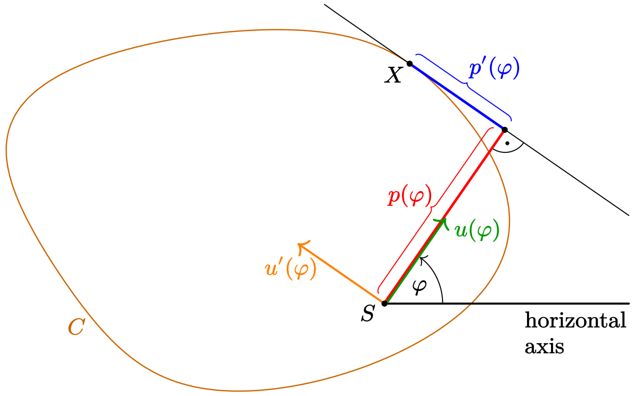

FIGURE 2 The support function p of the curve C.

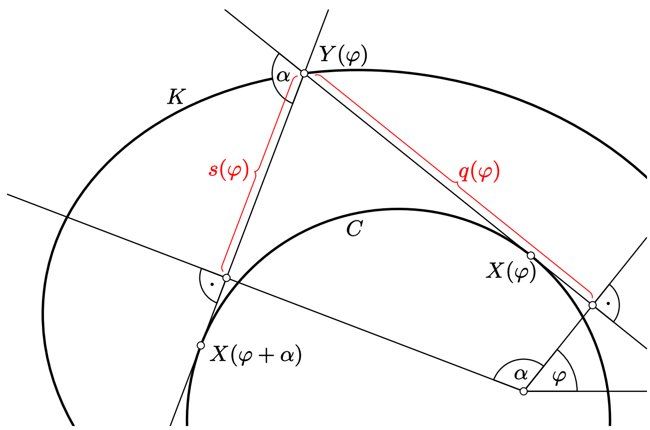

FIGURE 3 Construction of equiangular Poncelet pairs.

Theorem 6. *If C is a closed C^2 curve in the Euclidean plane with nonvanishing curvature and total curvature $2k\pi$, where $0 < k \in \mathbb{N}$, and $0 < \alpha < \pi$ is commensurable with π, then there exist $2k$ closed curves K_i, $i = 0, \dots, 2k-1$ such that (K_i, C) is an equiangular Poncelet pair with angle α.*

Proof. We can assume that C is parametrized by

$$C : [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto X(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi) \tag{2.1}$$

for a C^2 support function $p : [0, 2k\pi) \to \mathbb{R}$ as indicated in Figure 2. Here $2k\pi$ is the total curvature of C, $u(\varphi) = (\cos(\varphi), \sin(\varphi))$, and $\rho(\varphi) = p(\varphi) + p''(\varphi) > 0$ is the radius of curvature of C in the point $X(\varphi)$ (see, e.g., ref. [16]).

The condition that the circumscribed Poncelet polygon has external angle α translates into the equation

$$p(\varphi)u(\varphi) + q(\varphi)u'(\varphi) = p(\varphi+\alpha)u(\varphi+\alpha) + s(\varphi)u'(\varphi+\alpha) \tag{2.2}$$

for some function s (see Figure 3). By multiplying Equation (2.2) by $u(\varphi+\alpha)$, we get

$$q(\varphi) = \csc(\alpha)\big(p(\varphi+\alpha) - \cos(\alpha)p(\varphi)\big), \tag{2.3}$$

and hence

$$K : [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y(\varphi) = \csc(\alpha)\big(p(\varphi+\alpha)u'(\varphi) - p(\varphi)u'(\varphi+\alpha)\big) \tag{2.4}$$

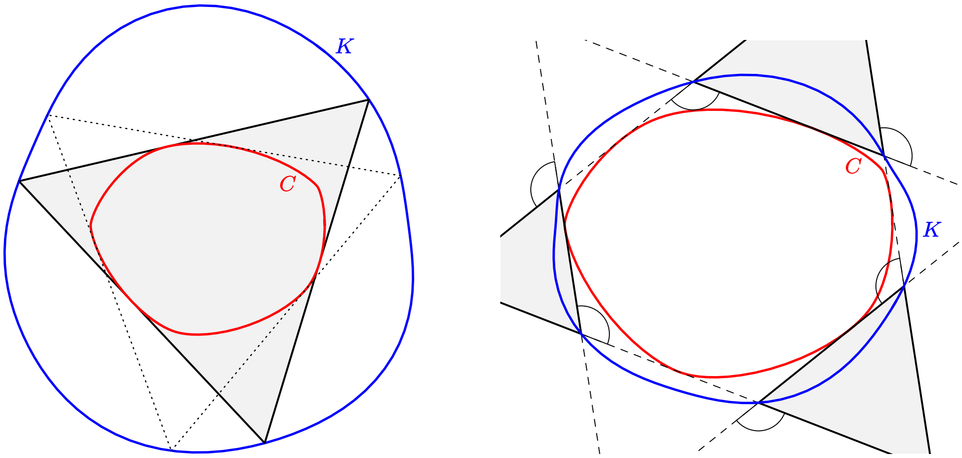

FIGURE 4 Illustration for Theorem 6. Solutions for $k = 1$: On the left an equiangular Poncelet triangle, shown in two positions. On the right an equiangular Poncelet hexagon. Both solutions have external angle $\alpha = 2\pi/3$. The hexagon can also be seen as a hexagon with external angle $\alpha = \pi/3$.

is an explicit C^2 parametrization of a possible vertex curve K. Note that Equation (2.4) also yields a vertex curve such that the Poncelet polygon has external angle α, if it is applied to the angle $\alpha_i = \alpha + i\pi$, $i = 0, \dots, 2k-1$. In this way, we obtain $2k$ Poncelet pairs (K_i, C), where

$$K_i\ :\ [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y_i(\varphi) = \csc(\alpha_i)\big(p(\varphi+\alpha_i)u'(\varphi) - p(\varphi)u'(\varphi+\alpha_i)\big). \tag{2.5}$$

See Figure 4 for an example with $k = 1, \alpha = 2\pi/3$. □

Remark 7. The Poncelet polygons in the Poncelet pair (K_i, C), $i = 0, \dots, 2k-1$, in Theorem 6 are given as follows: The Poncelet polygon with starting point $Y_i(\varphi) \in K_i$ has the vertices $Y_i(\varphi + j\alpha_i)$, where $j = 0, 1, \dots, m-1$. Here m is minimal such that $m\alpha_i$ is a multiple of $2k\pi$. This implies that this Poncelet polygon has

$$m = \frac{2\pi}{\alpha_i}\operatorname{lcm}\left(\frac{\alpha_i}{2\pi}, k\right) \tag{2.6}$$

vertices. We adopt the notation lcm for the least common multiple.

The vertex curves constructed in Theorem 6 may exhibit singularities. However, the following theorem shows that this only occurs in very special cases.

Theorem 8. *If the given envelope C has no self-intersections, then the vertex curve K_i in Theorem 6 is regular. If the given envelope C has self-intersections and all intersection angles are different from the angle α_i, the vertex curve is regular. A singularity of K can only occur in a self-intersection of C.*

Proof. If we take the derivative in Equation (2.5), we find

$$Y_i'(\varphi) = \csc(\alpha_i)\big(p'(\varphi+\alpha_i)u'(\varphi) - p(\varphi+\alpha_i)u(\varphi) - p'(\varphi)u'(\varphi+\alpha_i) - p(\varphi)u(\varphi+\alpha_i)\big).$$

We fix the parameter φ. By shifting the origin S to a point on the curve C and rotating the coordinate system, we may assume $\varphi = 0$ and $p(0) = p'(0) = 0$. Hence, a singularity $Y_i'(0) = 0$ can only occur, if $p(\alpha_i) = p'(\alpha_i) = 0$. □

An example of a singular vertex curve is shown in Figure 5.

Observe that the proof of Theorem 6 yields explicit formulas (2.1) and (2.4) for an equiangular Poncelet pair (K, C). A simple but particularly interesting family of examples is obtained by choosing $p(\varphi) = a + b\cos(\ell\varphi)$. Multiplying p by a constant results in a homothety of the curves C and K, hence we may assume $b = 1$ in the sequel. The condition

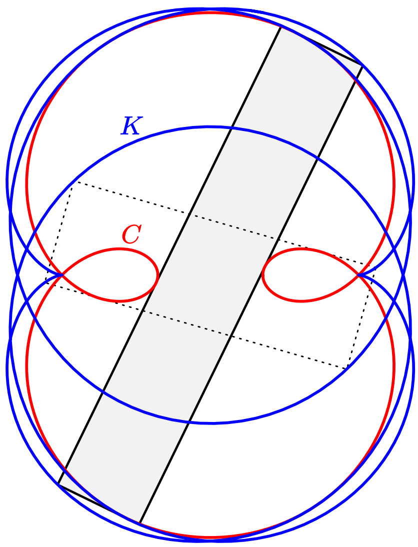

FIGURE 5 An example of a vertex curve K with singularities (cusps) for a Poncelet pair (C, K) which carries rectangles, that is, Poncelet polygons with exterior angle $\alpha = \pi/2$. The reason for the singularities is that the envelope C has self-intersections with intersection angle of the same size $\pi/2$. Note that here the extensions of the sides touch the envelope.

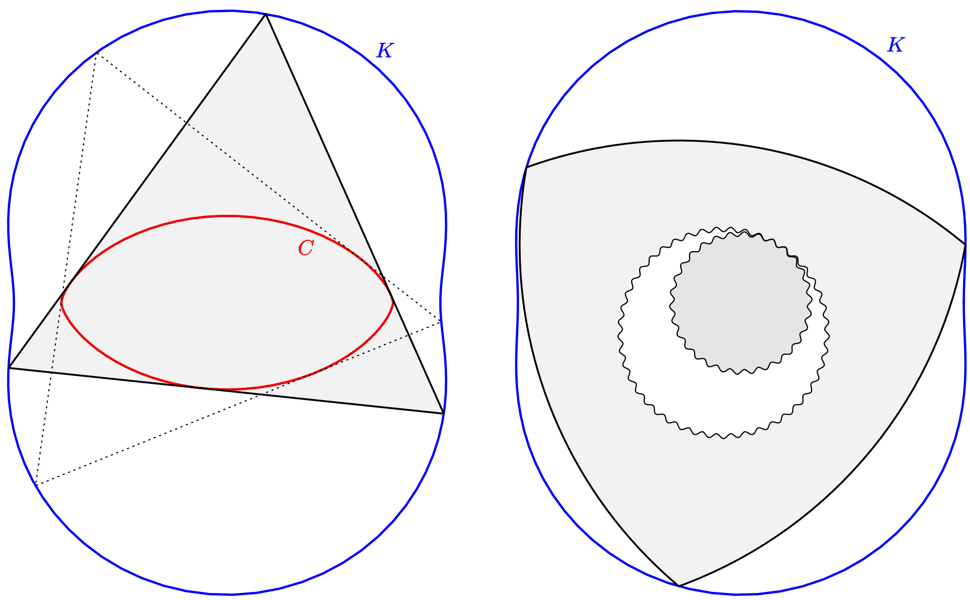

FIGURE 6 A Poncelet pair with $k = 1, n = 3, p(\varphi) = \frac{8}{5} + \frac{1}{2}\cos(2\varphi)$ on the left, the design of a Wankel engine on the right.

$p(\varphi) + p''(\varphi) > 0$ is satisfied if we choose $a > \ell^2 - 1$ if $\ell > 1$, and $a > 1 - \ell^2$ if $0 < \ell < 1$. In this situation, we get by Equation (2.4) for $\ell = 2, \alpha = \frac{2\pi}{3}$, the Poncelet pair

$$C : [0, 2\pi) \to \mathbb{R}^2, \ \ \varphi \mapsto u(\varphi)(a + \cos(2\varphi)) - 2u'(\varphi)\sin(2\varphi)$$

$$K : [0, 2\pi) \to \mathbb{R}^2, \ \ \varphi \mapsto 2au(\varphi) - u(3\varphi).$$

Hence, the vertex curve K is an epitrochoid. It turns out that the Poncelet triangle in this case is not only equiangular, but even equilateral. This is a special case of Theorem 9 below. Figure 6 shows on the left an example with $a = \frac{8}{5}$. Observe that for $a = 2 + \sqrt{3}$, we get the classical design of a Wankel engine (see Figure 6 on the right): Here, the constant a is chosen as the smallest value such that a Reuleaux triangle can turn inside the vertex curve K. In general, we have the following.

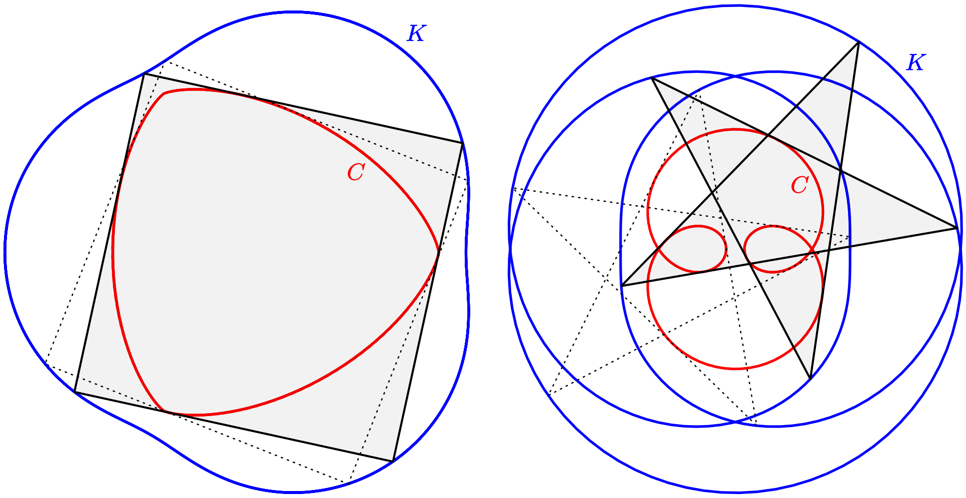

FIGURE 7 A Wankel engine with three combustion chambers with parameters $k = 1, \ell = 3, n = 4, a > \ell^2 - 1$ on the left, a regular pentagram as Poncelet polygon resulting from the parameters $k = 4, \ell = 2/3, n = l + 1, a = \cos(\alpha/2)n^2$ on the right.

Theorem 9. *Let* $0 < k \in \mathbb{N}, 0 < \ell \in \mathbb{Q}, \ell \neq \frac{2k}{m} - 1, m \in \mathbb{Z}, n = \ell + 1, \alpha = \frac{2k\pi}{n}$, *and* $p(\varphi) = a + \cos(\ell\varphi)$ *with* $a > \ell^2 - 1$ *if* $\ell > 1$, *and* $a > 1 - \ell^2$ *if* $0 < \ell < 1$. *Then the Poncelet polygon P for the Poncelet pair* (K, C) *given by Equations (2.1) and (2.4) is equiangular with angle* α *and equilateral with side length* $2a|\tan(\alpha/2)|$. *K is an epitrochoid. The midpoint of P rotates on a circle centered at the origin with radius 1.*

Proof. We first use $p(\varphi) = a + \cos(\ell\varphi)$ in the general formula (2.4). Unfortunately, this does not yield a very telling expression. However, if we replace φ by $\varphi - \alpha/2$ and observe that k is an integer, we get the reparametrization

$$K : [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y(\varphi) = a\sec(\alpha/2)u(\varphi) + (-1)^k u(n\varphi). \tag{2.7}$$

This shows that the vertex curve K is an epitrochoid. The vertices of the Poncelet polygon P are the points

$$\left\{Y(\varphi + j\alpha) = (-1)^k u(n\varphi) + a\sec(\alpha/2)u(\varphi + j\alpha), j = 1, \dots, \tfrac{n}{(n,k)}\right\}$$

where (n, k) denotes the greatest common divisor of n and k. Its midpoint, given by $(-1)^k u(n\varphi)$, rotates on the unit circle. The perimeter radius of P is $a|\sec(\alpha/2)|$ and the center angle over a side is α. Hence we get for the side length s of P

$$s = 2a|\sec(\alpha/2)\sin(\alpha/2)| = 2a\left|\tan(\alpha/2)\right|. \qquad \square$$

Figure 7 illustrates Theorem 9 with two examples. The first one is a Wankel engine with three combustion chambers. We note that in Glaeser et al. [17, p. 515ff], a variant of a Wankel engine design with a spherical triangular rotor turning in an equilateral spherical conic is discussed.

Remark 10. In Figure 7, on the right, the parameter a has been chosen minimal with the property that the curvature of the curve K does not change sign. Indeed, if $a = \cos(\alpha/2)n^2$, then the curvature of K is non-negative and there holds

$$\det\begin{pmatrix} Y'(\varphi) & Y''(\varphi)\end{pmatrix} = 2n^3(n+1)\begin{cases}\cos^2\left(\frac{\ell}{2}\varphi\right), & \text{if } k \text{ is odd}\\ \sin^2\left(\frac{\ell}{2}\varphi\right), & \text{if } k \text{ is even.}\end{cases}$$

and if $a > \cos(\alpha/2)n^2$, then the curvature of K is strictly positive.

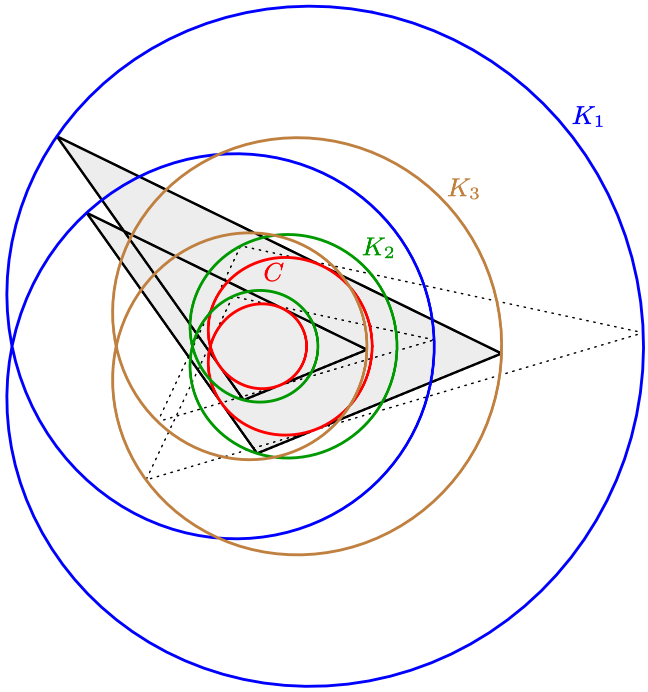

FIGURE 8 A Poncelet clan according to Corollary 11.

2.2 | Equiangular clans

Using the results from the previous section, it is now straightforward to construct equiangular Poncelet clans. Note that we generalize the definition of the term *equiangular* for clans in the following sense: We no longer require that all exterior angles of the Poncelet polygon are equal for all starting points, but only that the angle in each vertex does not depend on the starting point.

Corollary 11. *Let C be a closed C^2 curve in the Euclidean plane with nonvanishing curvature, given by Equation (2.1). Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be given external angles such that $\alpha_1 + \dots + \alpha_n = 2m\pi$, for a natural number $m \geqslant 1$. Then the curves $(C, K_1, K_2, \dots, K_n)$ form a Poncelet clan, where the vertex curve K_i is given by the parametrization*

$$K_i\ :\ [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y_i(\varphi) = \csc(\alpha_i)\big(p(\varphi + \alpha_i)u'(\varphi) - p(\varphi)u'(\varphi + \alpha_i)\big).$$

The polygon $P(\varphi)$ with vertices $Y_1(\varphi), Y_2(\varphi + \alpha_1), \dots, Y_n(\varphi + \alpha_1 + \dots + \alpha_{n-1}), Y_1(\varphi + 2\pi m), Y_2(\varphi + 2\pi m + \alpha_1), \dots, Y_n(\varphi + 2\pi m + \alpha_1 + \dots + \alpha_{n-1}), Y_1(\varphi + 4\pi m), \dots$ is a Poncelet polygon for each value of φ.

Figure 8 shows an example of such a Poncelet clan with three vertex curves K_1, K_2, K_3 and $k = 2$, $m = 1$.

Proof. The formula for the vertex curves K_i follows immediately from Equation (2.4). The polygon $P(\varphi)$ is by construction tangential to the envelope C and its vertices lie on $K_1, \dots, K_n$. □

Note that if the Poncelet polygon P is a triangle, then the triangles $P(\varphi)$ are similar for all φ.

Remark 12. Like in Theorem 6, each angle α_i in Corollary 11 can be replaced by

$$\alpha_{j_i} = \alpha_i + j_i\pi,$$

where $j_i = 0, \dots, 2k - 1$. Upon writing

$$K_i^{j_i}\ :\ [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y_i^{j_i}(\varphi) = \csc(\alpha_{j_i})\big(p(\varphi + \alpha_{j_i})u'(\varphi) - p(\varphi)u'(\varphi + \alpha_{j_i})\big)$$

we obtain all possible Poncelet clans with external angles $\alpha_1, \dots, \alpha_n$ as

$$(C, K_1^{j_1}, K_2^{j_2}, \dots, K_n^{j_n}),$$

where $j_i \in \{0, \dots, 2k-1\}$ leading to $(2k)^n$ Poncelet clans. For a fixed multi-index $\mathbf{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$ with entries in $\{0, \dots, 2k-1\}$, we can write down the vertices of the corresponding Poncelet polygon $P_{\mathbf{j}}(\varphi)$ as follows: Let l be the smallest natural number such that

$$l \sum_{i=1}^{n} \alpha_{j_i} = l \sum_{i=1}^{n} (\alpha_i + j_i \pi) = l\pi(2m + |\mathbf{j}|)$$

is a multiple of $2\pi m$, hence

$$l = \frac{\operatorname{lcm}(2m + |\mathbf{j}|, 2k)}{2m + |\mathbf{j}|}.$$

The vertices of $P_{\mathbf{j}}(\varphi)$ are obtained by going through the rows of the $(l \times n)$-matrix $(\mathbf{V}_{\ell\nu})_{\substack{1\leqslant \ell \leqslant l \\ 1 \leqslant \nu \leqslant n}}$ with entries

$$\mathbf{V}_{\ell\nu} = Y_\nu^{j_\nu}\left(\varphi + \pi(2m + |\mathbf{j}|)(\ell - 1) + \sum_{i=1}^{\nu-1} \alpha_{j_i}\right),$$

hence the number of vertices of the polygon is at most

$$\frac{n \cdot \operatorname{lcm}(2m + |\mathbf{j}|, 2k)}{2m + |\mathbf{j}|}. \tag{2.8}$$

Note that the formula (2.8) yields the correct number of vertices if all angles $\alpha_1, \dots, \alpha_n$ are pairwise distinct, which is generically the case, however, there are degenerate cases, where there are less vertices: For instance, if $\alpha_1 = \alpha_2 = \dots = \alpha_n = \frac{2\pi m}{n}$ and $\mathbf{j} = (i, i, \dots, i)$ for some $i \in \{0, \dots, 2k-1\}$, then all the curves of the corresponding Poncelet clan are identical and the situation reduces to the one in Theorem 6, where the formula (2.6) yields the correct number of vertices

$$\frac{n}{2m + in} \operatorname{lcm}\left(\frac{2m + 2ni}{n}, 2k\right),$$

whereas fomula (2.8) yields

$$\frac{n}{2m + in} \operatorname{lcm}(2m + 2ni, 2k).$$

3 | CONSTRUCTION OF GENERAL PONCELET PAIRS AND CLANS

In this section, we will construct general Poncelet pairs and clans. In Sections 3.2 and 3.3, we specify the vertex curve K and find corresponding envelopes, while in Sections 3.4 and 3.5, we specify the envelope and find corresponding vertex curves. The necessary tools are provided in Section 3.1.

3.1 | Torsion maps of S^1

In order to use standard terminology and to avoid the unnecessary appearance of the factor 2π throughout, we will use the identification $S^1 \cong \mathbb{R}/\mathbb{Z}$ in this section.

The following construction of Poncelet pairs will make use of diffeomorphisms $f : S^1 \to S^1$ with the property that the nth iterate of f equals the identity on S^1, that is, $f^0 = f^n = \mathrm{id}_{S^1}$, but $f^i \neq \mathrm{id}_{S^1}$ for $0 < i < n$. We will call such a map f a *torsion map* and we will, without loss of generality, restrict ourselves to orientation preserving diffeomorphisms.

As an example of a torsion map, consider the rotation $r_\alpha : S^1 \to S^1$ about the angle $\alpha = \frac{m}{n} \in \mathbb{Q}$, where $\frac{m}{n}$ is a reduced fraction. Conjugating r_α by any diffeomorphism $h : S^1 \to S^1$ yields another torsion map: If $f = h^{-1} \circ r_\alpha \circ h$, then $f^n = h^{-1} \circ r_\alpha^n \circ h = \mathrm{id}_{S^1}$.

We will now show that all torsion maps f arise from rotations up to conjugation by diffeomorphisms of S^1.

Every orientation preserving diffeomorphism $f : S^1 \to S^1$ lifts to an increasing diffeomorphism $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f \circ \pi$, where $\pi : \mathbb{R} \to S^1, x \mapsto x \mod 1$ is the usual quotient map and $F(x+1) = F(x) + 1$ for all $x \in \mathbb{R}$. Poincaré introduced the rotation number of f as

$$\tau(f) = \lim_{k\to\infty} \frac{F^k(x) - x}{k} \mod 1$$

and showed that this limit always exists and that it is independent of the lift and $x \in \mathbb{R}$ (see ref. [18]).

To proceed, we now need the following result. A variant of it appears, for instance, in ref. [19, Ex. 14 (b)]. For the readers convenience, we provide an explicit proof.

Proposition 13. *Let $f : S^1 \to S^1$ be a torsion map, that is, an orientation preserving diffeomorphism such that $f^n = \mathrm{id}_{S^1}$ and $f^i \neq \mathrm{id}_{S^1}$ for $0 < i < n$. Then we have:*

1. *The rotation number of f equals $\frac{m}{n}$ for some $m \in \mathbb{Z}$ and the fraction $\frac{m}{n}$ is reduced.*
2. *f is conjugate to the rotation $r_{\frac{m}{n}}$ about the angle $\frac{m}{n}$, that is, there is an orientation preserving diffeomorphism $h : S^1 \to S^1$ such that $f = h^{-1} \circ r_{\frac{m}{n}} \circ h$.*

Proof. For the first item, consider the unique lift F of f such that $F(0) \in [0, 1)$: If $0 \leqslant x < 1$, we find inductively $F^i(x) < i + 1$ and since $f^n = \mathrm{id}_{S^1}$, we have $F^n(x) = x + m$ for some $m \in \mathbb{Z}$ and $m \leqslant n$. If $k = nd$ for some $d \in \mathbb{N}$, it follows that

$$\frac{F^k(x) - x}{k} = \frac{x + md - x}{nd} = \frac{m}{n}$$

and therefore $\tau(f) = \frac{m}{n}$ (or, if $m = n = 1$, $\tau(f) = 0$ in which case $f = \mathrm{id}_{S^1}$).

If the rotation number of an orientation preserving diffeomorphism is $\frac{m}{n} \in \mathbb{Q}$, where $\frac{m}{n}$ is reduced, then every periodic orbit of f has period n. Therefore, $\frac{m}{n}$ is reduced as the minimal length of a period of f is n by assumption [20, p. 12].

For the second item, we will construct a diffeomorphism $h : S^1 \to S^1$, which conjugates f to the rotation $r_{\frac{m}{n}}$ explicitly: The rotation $r_{\frac{m}{n}} : S^1 \to S^1$ lifts to the increasing diffeomorphism $R_{\frac{m}{n}} : \mathbb{R} \to \mathbb{R}, x \mapsto x + \frac{m}{n}$. Define the increasing diffeomorphism $H : \mathbb{R} \to \mathbb{R}$,

$$H = \frac{1}{n} \sum_{j=0}^{n-1} F^j.$$

By construction, if $F \in C^k$, then $H \in C^k$ as well and $H \circ F = R_{\frac{m}{n}} \circ H$. Since $H(x + 1) = H(x) + 1$, the map H descends to an orientation preserving diffeomorphism $h : S^1 \to S^1$.

Since $F = H^{-1} \circ R_{\frac{m}{n}} \circ H$, we obtain

$$\pi \circ F = \pi \circ H^{-1} \circ R_{\frac{m}{n}} \circ H = h^{-1} \circ r_{\frac{m}{n}} \circ h \circ \pi$$

so that F is a lift of $h^{-1} \circ r_{\frac{m}{n}} \circ h$ and we conclude that $f = h^{-1} \circ r_{\frac{m}{n}} \circ h$. □

Before we proceed to construction of Poncelet pairs using Proposition 13, we remark that the connection between conjugacy to a rotation about a rational angle and Poncelet's theorem already appears in ref. [21, Lemma 1.3], where Poncelet's theorem is recast in a measure theoretic framework for the case where C and K are nested ellipses. In a different context, exploiting dynamical billiards of an ellipse, the connection between a rational rotation number and periodicity of a billiard trajectories is established in ref. [22]. Also in the context of billiard dynamics, for a given circle K, Lopes and Sebastiani [23] construct a family of nested circles C such that (K, C) is a Poncelet pair. This article also provides a formula that counts the number of such Poncelet pairs, where the Poncelet polygon closes after n steps.

3.2 | Poncelet pairs for given vertex curve K

In order to construct Poncelet pairs, we will change back to the convention $S^1 \cong \mathbb{R}/(2\pi\mathbb{Z})$ in all that follows.

Let $K : S^1 \to \mathbb{R}^2, \varphi \mapsto Y(\varphi)$, be a positively oriented regular C^k curve, $k \geqslant 2$, with nonvanishing curvature and let $f \in C^k(S^1, S^1)$ be a torsion map such that $f^0 = f^n = \mathrm{id}_{S^1}$, where $n \geqslant 2$. According to the discussion in the previous section, there exists a C^k diffeomorphism $h : S^1 \to S^1$ such that

$$f = h^{-1} \circ r_{\frac{2\pi m}{n}} \circ h,$$

where $n > m \in \mathbb{N}$. For each fixed $\varphi \in S^1$, the function f gives rise to a polygon P with vertices

$$Y(\varphi), Y(f(\varphi)), \dots, Y(f^{n-1}(\varphi))$$

and its edges are the segments between $Y(f^i(\varphi))$ and $Y(f^{i+1}(\varphi))$, where i is taken mod n. For fixed φ, we consider the line

$$s \mapsto E(s, \varphi) = (1-s)Y(\varphi) + sY(f(\varphi))$$

connecting two consecutive edges of the polygon P. The envelope $C : S^1 \to \mathbb{R}^2, \varphi \mapsto X(\varphi)$, of these lines when φ varies is obtained by requiring that $X(\varphi) = E(s(\varphi), \varphi)$ is tangential to $\partial_s E(s, \varphi)$ for all $\varphi \in S^1$. This condition translates formally into

$$s = -\frac{\langle Y', J(Y \circ f - Y)\rangle}{\langle (Y \circ f - Y)', J(Y \circ f - Y)\rangle}, \tag{3.1}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Hence, the envelope is a regular C^{k-1} curve $C : S^1 \to \mathbb{R}^2$, given by

$$X = Y - \frac{\langle Y', J(Y \circ f - Y)\rangle}{\langle (Y \circ f - Y)', J(Y \circ f - Y)\rangle}(Y \circ f - Y), \tag{3.2}$$

whenever s has no singularities. If $0 \leqslant s \leqslant 1$, then the contact point of the line $E(s, \varphi)$ with the envelope C lies on the segment joining the two vertices $Y(\varphi)$ and $Y(f(\varphi))$, otherwise on its prolongation.

Since $f = h^{-1} \circ r_\alpha \circ h$ with $0 < \alpha \notin 2\pi\mathbb{N}$ being a rational multiple of 2π, we consider $Z = Y \circ h^{-1}$. With the notation $t = h(\varphi)$ and $\Delta(t) = Z(t + \alpha) - Z(t)$, the formula for X as a function of t becomes

$$X = Z - \frac{\langle Z', J\Delta\rangle}{\langle \Delta', J\Delta\rangle}\Delta. \tag{3.3}$$

Here, X is well-defined if and only if $\langle \Delta', J\Delta\rangle \neq 0$ and in this case, $X \in C^{k-1}(S^1, \mathbb{R}^2)$ since Z is of class C^k.

Proposition 14.

1. *Suppose that X is well-defined. Then it is a regular curve, that is, $X' \neq 0$, if and only if*

$$0 \neq \det \begin{pmatrix} \langle Z', J\Delta\rangle & \langle Z'_\alpha, J\Delta\rangle \\ \langle Z'', J\Delta\rangle + 2\langle Z', J\Delta'\rangle & \langle Z''_\alpha, J\Delta\rangle + 2\langle Z'_\alpha, J\Delta'\rangle \end{pmatrix},$$

where $Z_\alpha(t) = Z(t + \alpha)$.

2. *If Z is of class C^3, X is regular if and only if the curvature of X is never zero.*

Proof. We have $\langle X', J\Delta\rangle = 0$. Hence X is regular if and only if $\langle X', J\Delta'\rangle \neq 0$, since Δ and Δ' are linearly independent. Multiplying this condition by $\langle \Delta', J\Delta\rangle \neq 0$ yields the equivalent condition

$$0 \neq \langle Z'_\alpha, J\Delta\rangle\big(\langle Z'', J\Delta\rangle + 2\langle Z', J\Delta'\rangle\big) - \langle Z', J\Delta\rangle\big(\langle Z''_\alpha, J\Delta\rangle + 2\langle Z', J\Delta'\rangle\big),$$

which can be restated as claimed, since $\langle Z'_\alpha, J\Delta'\rangle = \langle Z', J\Delta'\rangle$.

If Z is of class C^3, X is of class C^2. If $X' \neq 0$, it follows from $\langle X', J\Delta\rangle = 0$ that

$$X' = \underbrace{\tfrac{\langle X', \Delta\rangle}{\|\Delta\|^2}}_{=:g} \Delta$$

where $g \in C^1(S^1)$ never vanishes. Hence

$$\langle X'', JX'\rangle = \langle g'\Delta + g\Delta', gJ\Delta\rangle = g^2\langle \Delta', J\Delta\rangle \neq 0,$$

since X is well-defined. Conversely, if $\langle X'', JX'\rangle \neq 0$, X' cannot vanish. □

The next theorem is the main result of this section. It shows how to construct a Poncelet pair (K, C) if the vertex curve K and a torsion map $f : S^1 \to S^1$ are given.

Theorem 15. *Let $K : S^1 \to \mathbb{R}^2$, $\varphi \mapsto Y(\varphi)$, be a positively oriented regular C^k curve, $k \geqslant 2$, with nonvanishing curvature. Then the pair (K, C), where the envelope $C \in C^{k-1}(S^1, \mathbb{R}^2)$ is defined via Equation (3.2), is a Poncelet pair. If $k \geqslant 3$ and the curvature of C does not vanish, then C is a regular curve. If K has no self-intersections, and $k \geqslant 2$, then the function s defined in Equation (3.1) satisfies $0 < s < 1$, that is, the points of contact of the Poncelet polygon with the envelope C lie inside the sides.*

Proof. Only the last part of the claim remains to be proven. So, assume that K has no self-intersections. We parametrize C via Equation (3.3) and show that $\langle \Delta', J\Delta\rangle > 0$, which implies that C is well-defined. Since Z has no self-intersections, it parametrizes the boundary of a convex set and hence it lies entirely on one side of any of its tangents. This together with the fact that Z is positively oriented and has nonvanishing curvature translates into the fact that for any $0 < \alpha < 2\pi$, we have

$$\langle \Delta(t), JZ'(t)\rangle > 0 \text{ and } \langle -\Delta(t), JZ'(t+\alpha)\rangle > 0. \tag{3.4}$$

Adding the two inequalities gives $0 < \langle \Delta(t), J(Z'(t) - Z'(t+\alpha))\rangle = \langle \Delta', J\Delta\rangle$.

We are left to show that $s \in (0,1)$, where $s = -\frac{\langle Z', J\Delta\rangle}{\langle \Delta', J\Delta\rangle}$. According to the first inequality of Equation (3.4), we have $-\langle Z', J\Delta\rangle = \langle JZ', \Delta\rangle > 0$, so that $s > 0$. The condition $s < 1$ is equivalent to

$$-\langle Z'(t), J\Delta(t)\rangle < \langle \Delta'(t), J\Delta(t)\rangle \Longleftrightarrow 0 < \langle Z'(t+\alpha), J\Delta(t)\rangle,$$

which holds true by the second inequality of Equation (3.4). □

The next result states that a given Poncelet pair (K, C) induces a torsion map f.

Proposition 16. *Let (K, C) be a Poncelet pair for Poncelet n-gons. Assume that K is a C^2 curve parametrized by $\gamma : S^1 \to \mathbb{R}^2$ with $\dot\gamma \neq 0$, and C a C^2 curve with nonvanishing curvature. Let $P_1 = \gamma(t_1), P_2 = \gamma(t_2)$ be two consecutive vertices of a Poncelet polygon. Then the map $f : S^1 \to S^1, t_1 \mapsto t_2$, constitutes an orientation preserving C^2 diffeomorphism with $f^n = \mathrm{id}_{S^1}$ and $f^i \neq \mathrm{id}_{S^1}$ for $0 < i < n$.*

Proof. By applying a projective map if necessary, we may assume the situation depicted in Figure 9.

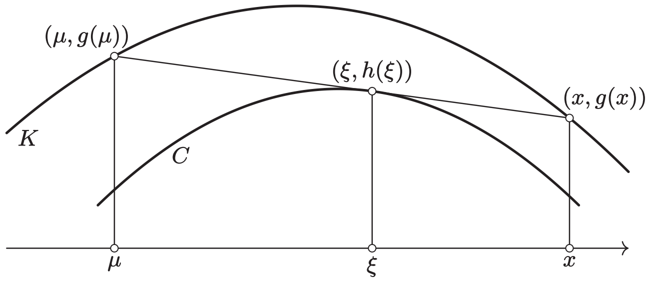

FIGURE 9 Construction of the diffeomorphism $f : S^1 \to S^1$.

Here, the vertex curve K is locally given as the graph of a C^2 function g, and the envelope C as graph of a C^2 function h. We are interested in the regularity of the map $x \mapsto \mu$, as defined in Figure 9. We have

$$g(x) - h(\xi) = h'(\xi)(x - \xi)$$
$$h(\xi) - g(\mu) = h'(\xi)(\xi - \mu).$$

Consider the C^1 map

$$F : \mathbb{R}^{1+2} \to \mathbb{R}^2, \quad (x, \xi, \mu) \mapsto \begin{pmatrix} g(x) - h(\xi) - h'(\xi)(x - \xi) \\ h(\xi) - g(\mu) - h'(\xi)(\xi - \mu) \end{pmatrix}$$

The Jacobian matrix of F is given by

$$DF(x, \xi, \mu) = \begin{pmatrix} g'(x) - h'(\xi) & (\xi - x)h''(\xi) & 0 \\ 0 & (\mu - \xi)h''(\xi) & h'(\xi) - g'(\mu). \end{pmatrix}$$

In particular, we have

$$\det \begin{pmatrix} (\xi - x)h''(\xi) & 0 \\ (\mu - \xi)h''(\xi) & h'(\xi) - g'(\mu) \end{pmatrix} = (\xi - x)h''(\xi)(h'(\xi) - g'(\mu)) \neq 0$$

and hence the implicit function theorem guarantees the local existence of a C^1 function $x \mapsto (\xi(x), \mu(x))$ such that $F(x, \xi(x), \mu(x)) = 0$, locally. However, for the partial function $\mu(x)$, we get a better regularity, namely C^2, since

$$\begin{pmatrix} \xi'(x) \\ \mu'(x) \end{pmatrix} = -\begin{pmatrix} (\xi(x) - x)h''(\xi(x)) & 0 \\ (\mu(x) - \xi(x))h''(\xi(x)) & h'(\xi(x)) - g'(\mu(x)) \end{pmatrix}^{-1} \begin{pmatrix} g'(x) - h'(\xi(x)) \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{g'(x)-h'(\xi(x))}{(x-\xi(x))h''(\xi(x))} \\ \frac{\mu(x)-\xi(x))(g'(x)-h'(\xi(x)))}{(x-\xi(x))(g'(\mu(x))-h'(\xi(x)))} \end{pmatrix}.$$

The conversion to the parameter t of the parametrization $\gamma : S^1 \to \mathbb{R}^2$ does not change this regularity. Indeed, if $\gamma(t_1) = (x, g(x))$ and $\gamma(t_2) = (\mu, g(\mu))$, we have $t_2 = \gamma_1^{-1}(\mu(\gamma_1(t_1)))$.

With this regularity established, the properties of the function f follow from the properties of the Poncelet polygon in (K, C). □

Remark 17. Notice that according to Proposition 16, Theorem 15 describes the general case of a Poncelet pair (K, C) if K is a C^2 curve, and C a C^2 curve with nonvanishing curvature. Indeed, let (K, C) be such a pair, where K is given by a positively oriented parametrization, then, by Proposition 16, this defines an orientation preserving C^2 diffeomorphism $f : S^1 \to S^1$ with $f^n = \mathrm{id}_{S^1}$, that is, a torsion map, such that the curve given by the parametrization (3.2) is the original envelope C.

In Section 2.1, we have constructed equiangular Poncelet pairs for a given envelope C. Now, we want to consider equiangular pairs, if the vertex curve K is given. Let $m, n \in \mathbb{N}$ be coprime, $n \geqslant 2$ and consider again a vertex curve K with positive curvature given by the support function $p(\varphi) = a + \cos(\ell\varphi)$, where $\ell = \frac{n}{m}$. The condition for positive curvature is given by $a > \ell^2 - 1$ if $\ell > 1$ and $a > 1 - \ell^2$ if $0 < \ell < 1$. In this case, K can be parametrized by

$$K : [0, 2\pi m) \to \mathbb{R}^2, \varphi \mapsto Y(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi). \tag{3.5}$$

If $\alpha = 2\pi/\ell$, then the function $f(\varphi) = \varphi + \alpha \mod 2\pi$ is a smooth torsion map as $f^n = \mathrm{id}_{S^1}$ and $f^i \neq \mathrm{id}_{S^1}$ for $0 < i < n$. The envelope of the segments of the Poncelet polygon P with vertices

$$Y(t + j\alpha), j = 0, \dots, n - 1$$

is given by

$$C : [0, 2\pi m) \to \mathbb{R}^2, \varphi \mapsto X(\varphi) = Y(\varphi) + s(\varphi)(Y(\varphi + \alpha) - Y(\varphi)), \tag{3.6}$$

where

$$s(\varphi) = \frac{1}{2}\left(1 + \cot^2\left(\frac{\pi}{\ell}\right)\frac{p'(\varphi)}{p(\varphi)}\right).$$

Note that s is well-defined, provided $a > 1$ and in this case, the pair (K, C) given by Equations (3.5) and (3.6) is an equiangular Poncelet pair with angle α.

Note that the polygon collapses to a multiplicity 2 segment if $n = 2$ and in this case, the curve C degenerates to a single point since $X(\varphi) = (0, 0)$. We will therefore now assume $n \geqslant 3$: If $(1 + a - 2\ell^2)^2 > 0$, then the curvature of X is positive and X is therefore a regular curve provided $a > \max\{2\ell^2 - 1, 1\}$.

If $\ell \in \mathbb{N}$ (and hence $\ell \geqslant 3$), P is a regular ℓ-gon and if ℓ is odd, the curve Y is the boundary of a body of constant width as

$$p(\varphi + \pi) + p(\varphi) \equiv 2a.$$

Summarizing, we have:

Proposition 18. *Let $m, n \in \mathbb{N}$ be coprime, $n \geqslant 3$, $\ell = \frac{n}{m}$, $\alpha = \frac{2\pi}{\ell}$ and $p(\varphi) = a + \cos(\ell\varphi)$, where $a > \max\{\ell^2 - 1, 1\}$. Then the Poncelet polygon P for the Poncelet pair (K, C) given by Equations (3.5) and (3.6) is equiangular with angle α and K is a curve with positive curvature. We furthermore have:*

1. *If $a > \max\{2\ell^2 - 1, 1\}$, the curvature of C is positive.*
2. *If $\ell \in \mathbb{N}$, then P is a regular ℓ-gon and the condition for positive curvature of C reduces to $a > 2\ell^2 - 1$.*
3. *If $\ell \in \mathbb{N}$ is odd, then K is the boundary of a body with constant width.*

An example illustrating Proposition 18 is shown in Figure 10.

Remark 19. If $s(\varphi) \in [0, 1]$, then the segments of P rather than their prolongation touch C. This condition translates into $\left|\cot\left(\frac{\pi}{\ell}\right) p'(\varphi)/p(\varphi)\right| \leqslant 1$. Since $|p'/p| \leqslant \frac{\ell}{\sqrt{a^2-1}}$, it is sufficient to require that

$$a \geqslant \sqrt{\cot^2\left(\frac{\pi}{\ell}\right)\ell^2 + 1}. \tag{3.7}$$

If $\ell \in \mathbb{N}$, and the condition $a > 2\ell^2 - 1$ for positive curvature of C is satisfied, then Equation (3.7) is automatically met since in this case, $2\ell^2 - 1 > \sqrt{\cot^2\left(\frac{\pi}{\ell}\right)\ell^2 + 1}$.

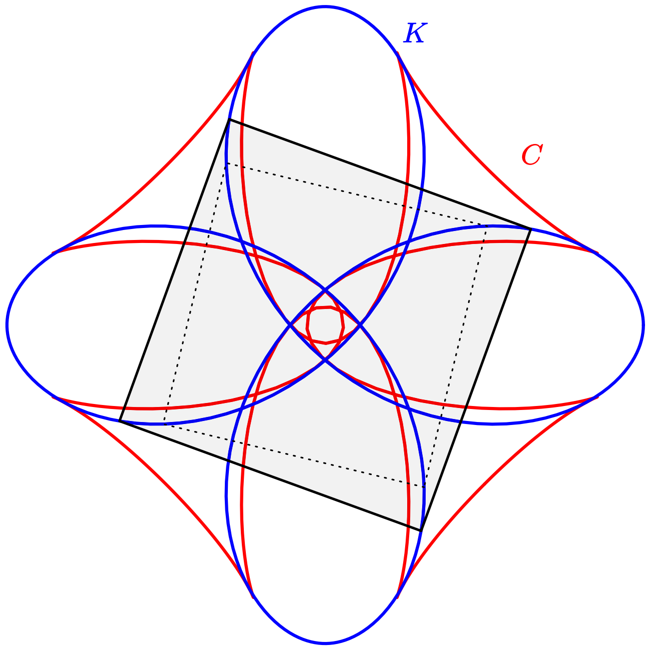

FIGURE 10 The curves K and C obtained in Proposition 18 with $m = 3$, $n = 4$, and $a = \frac{53}{45}$. Note that here the extensions of the sides of the Poncelet square touch the envelope C.

3.3 | Poncelet clans for given vertex curve K

Let again $K : S^1 \to \mathbb{R}^2, \varphi \mapsto Y(\varphi)$, be a positively oriented regular C^2 curve with nonvanishing curvature. This time, we consider orientation preserving diffeomorphisms $f_i : S^1 \to S^1, 1 \leqslant i \leqslant n$, where $n \geqslant 3$ such that

$$f_n = (f_{n-1} \circ \cdots \circ f_1)^{-1}.$$

We will henceforth write $g_i = f_i \circ f_{i-1} \circ \ldots \circ f_1$ for $1 \leqslant i \leqslant n$ and define $g_0 = \mathrm{id}_{S^1}$. For the following construction, we require that the Poncelet polygon with vertices $Y(g_i(\varphi)), i \in \{0, \ldots, n-1\}$, is not degenerate for all $\varphi \in S^1$. By this, we mean that the vertices are always different from each other. This condition can be reformulated as follows: If two vertices of the polygon coincide, then we must have $g_j(\varphi_0) = g_i(\varphi_0)$ for some $\varphi_0 \in S^1$ and $i \neq j$. We may assume without loss of generality that $i < j$ so that $g_i(\varphi_0)$ must be a fixed point of $g_j \circ g_i^{-1}$. Therefore, we will require $g_j \circ g_i^{-1}$ to not have fixed points for all $1 \leqslant i < j \leqslant n$. The curve $C_i : S^1 \to \mathbb{R}^2$ is then given by the envelope of the segments $(1-s)Y(g_{i-1}(\varphi)) + sY(g_i(\varphi))$, where i is taken mod n. According to Equation (3.2), C_i will be parametrized by

$$X_i = Y \circ g_{i-1} - \frac{\langle (Y \circ g_{i-1})', J(Y \circ g_i - Y \circ g_{i-1}) \rangle}{\langle (Y \circ g_i - Y \circ g_{i-1})', J(Y \circ g_i - Y \circ g_{i-1}) \rangle} (Y \circ g_i - Y \circ g_{i-1}),$$

and $(K, C_1, \ldots, C_n)$ is a Poncelet clan.

3.4 | Poncelet pairs for given envelope C

Here, we use the same approach as in Section 2. Let the envelope C be given by Equation (2.1). Consider an orientation preserving diffeomorphism $f \in C^2(S^1_k, S^1_k)$, where $S^1_k = \mathbb{R}/2k\pi\mathbb{Z}$ such that $f^0 = f^n = \mathrm{id}_{S^1_k}$, but $f^i \neq \mathrm{id}_{S^1_k}$ for $0 < i < n$ with $n > 2$. Then consider $X(\varphi)$ and $X(f(\varphi))$ as points of contact of two consecutive sides of a Poncelet polygon with the given curve C. The geometric situation is as in Figure 3, but with $f(\varphi)$ in place of $\varphi + \alpha$. Then the point $Y(\varphi) = p(\varphi)u(\varphi) + q(\varphi)u'(\varphi)$ on the vertex curve K satisfies

$$p(\varphi)u(\varphi) + q(\varphi)u'(\varphi) = p(f(\varphi))u(f(\varphi)) + s(\varphi)u'(f(\varphi)). \tag{3.8}$$

As in Section 2, we find

$$q(\varphi) = \frac{p(f(\varphi)) - p(\varphi)\langle u(\varphi), u(f(\varphi))\rangle}{\langle u'(\varphi), u(f(\varphi))\rangle}.$$

Hence we have the following result.

Theorem 20. *Let C be a closed C^2 curve in the Euclidean plane with nonvanishing curvature, given by Equation (2.1), and $f \in C^2(S^1_k, S^1_k)$ an orientation preserving diffeomorphism such that $f^0 = f^n = \mathrm{id}_{S^1_k}$, but $f^i \neq \mathrm{id}_{S^1_k}$ for $0 < i < n$ with $n > 2$. Suppose that $\langle u'(\varphi), u(f(\varphi))\rangle \neq 0$ for all φ. Then (K, C) is a Poncelet pair for the vertex curve*

$$K : [0, 2k\pi) \to \mathbb{R}^2, \quad \varphi \mapsto Y(\varphi) = p(\varphi)u(\varphi) + \frac{p(f(\varphi)) - p(\varphi)\langle u(\varphi), u(f(\varphi))\rangle}{\langle u'(\varphi), u(f(\varphi))\rangle} u'(\varphi).$$

3.5 | Poncelet clans for given envelope C

Let $C : S^1_k \to \mathbb{R}^2$ be a positively oriented regular C^2 curve with nonvanishing curvature given by a support function p as $X(\varphi) = p(\varphi)u(\varphi) + p'(\varphi)u'(\varphi)$ and consider nontrivial orientation preserving C^2 diffeomorphisms $f_i : S^1_k \to S^1_k$, $1 \leqslant i \leqslant n$, where $n \geqslant 3$ such that

$$f_n = (f_{n-1} \circ \cdots \circ f_1)^{-1}$$

and we will write again $g_i = f_i \circ f_{i-1} \circ \ldots \circ f_1$ for $1 \leqslant i \leqslant n$ and define $g_0 = \mathrm{id}_{S^1_k}$. Imitating the construction given in Theorem 20, if $\langle u'(g_{i-1}(\varphi)), u(g_i(\varphi)\rangle \neq 0$ for all φ and $i \in \{1, \ldots, n\}$, we obtain for all $i \in \{1, \ldots, n\}$, a C^2 vertex curve $K_i : S^1_k \to \mathbb{R}^2$ defined by

$$\varphi \mapsto Y_i(\varphi) = p(g_{i-1}(\varphi))u(g_{i-1}(\varphi)) + \frac{p(g_i(\varphi)) - p(g_{i-1}(\varphi))\langle u(g_{i-1}(\varphi)), u(g_i(\varphi))\rangle}{\langle u'(g_{i-1}(\varphi)), u(g_i(\varphi))\rangle} u'(g_{i-1}(\varphi)).$$

The polygon $P(\varphi)$ with vertices $Y_1(\varphi), Y_2(\varphi), \ldots, Y_n(\varphi)$ is then a Poncelet polygon for each value of φ.

ACKNOWLEDGMENTS

We would like to thank the referees for their valuable remarks which helped to improve this article.

Open access publishing facilitated by Eidgenossische Technische Hochschule Zurich, as part of the Wiley - Eidgenossische Technische Hochschule Zurich agreement via the Consortium Of Swiss Academic Libraries.

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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How to cite this article: Hungerbühler, N., Wasem, M.: Poncelet curves. Z Angew Math Mech. 105, e70225 (2025). https://doi.org/10.1002/zamm.70225