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Non-linear elliptic systems with measure-valued right hand side

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1 Introduction

We consider existence and compactness questions for elliptic systems of the form

$$-\text{div}\ \sigma(x, u(x), Du(x)) = \mu \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

with measure-valued right hand side on an open, bounded domain Ω in $\mathbb{R}^n$. We assume that σ satisfies the following hypotheses (H0)-(H3). Here $\mathbb{M}^{m\times n}$ denotes the space of real $m \times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (we use the usual summation convention) and the tensor product $a \otimes b$ of two vectors $a, b \in \mathbb{R}^m$ is defined to be the matrix $(a_i b_j)_{i,j=1,\ldots,m}$.

(H0) (continuity) $\sigma\colon \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e., $x \mapsto \sigma(x, u, p)$ is measurable for every (u, p) and $(u, p) \mapsto \sigma(x, u, p)$ is continuous for almost every $x \in \Omega$.

(H1) (monotonicity) For all $x \in \Omega$, $u \in \mathbb{R}^m$ and all $F, G \in \mathbb{M}^{m\times n}$ there holds

$$(\sigma(x, u, F) - \sigma(x, u, G)) : (F - G) \geq 0.$$

(H2) (coercivity and growth) There exist constants $c_1, c_3 > 0$, $c_2 \geq 0$ and p, q with $1 < p < n$ and $q - 1 < \frac{n}{n-1}(p - 1)$ such that for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F \in \mathbb{M}^{m\times n}$

$$\begin{aligned} \sigma(x, u, F) : F &\geq c_1|F|^p - c_2, \\ |\sigma(x, u, F)| &\leq c_3|F|^{q-1} + c_3. \end{aligned}$$

(H3) (structure condition) For all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F \in \mathbb{M}^{m\times n}$ there holds

$$\sigma(x, u, F) : MF \geq 0$$

for all matrices $M \in \mathbb{M}^{m\times m}$ of the form $M = \text{Id} - a \otimes a$ with $|a| \leq 1$.

Remarks. 1) Assumption (H0) ensures that $\sigma(x, u(x), U(x))$ is measurable on Ω for measurable functions $u : \Omega \to \mathbb{R}^m$ and $U : \Omega \to \mathbb{M}^{m\times n}$.

2) A typical example for a function σ satisfying (H3) is $\sigma(x, u, p) = \alpha(x, u, p)p$ with a real valued non-negative function α.

A serious technical obstacle is that for $p \in (1, 2 - \frac{1}{n}]$ solutions of the system (1.1) in general do not belong to the Sobolev space $W^{1,1}$. This fact has led to the use of renormalized solutions in [LM] and generalized entropy solutions in [BB] for elliptic equations of the above type (see also [Le]). We will use a notion of solution where the weak derivative Du is replaced by the approximate derivative ap Du. Recall that a measurable function u is said to be approximately differentiable at $x \in \Omega$ if there exists a matrix $F_x \in \mathbb{M}^{m\times n}$ such that for all $\varepsilon > 0$

$$\lim_{r\to 0} \frac{1}{r^n} \operatorname{meas} \{y \in B(x, r) \colon |u(y) - u(x) - F_x(y - x)| > \varepsilon r\} = 0.$$

We write ap $Du(x) = F_x$.

Definition 1 *A measurable function $u : \Omega \to \mathbb{R}^m$ is called a solution of the system (1.1) if*

(i) *u is almost everywhere approximately differentiable,*
(ii) *$\eta \circ u \in W^{1,1}(\Omega; \mathbb{R}^m)$ for all $\eta \in C^1_0(\mathbb{R}^m; \mathbb{R}^m)$,*
(iii) *$\sigma(\cdot, u, \mathrm{ap}\ Du) \in L^1(\Omega; \mathbb{M}^{m\times n})$,*
(iv) *the equation*

$$-\operatorname{div} \sigma(x, u(x), \mathrm{ap}\ Du(x)) = \mu$$

holds in the sense of distributions.

Moreover we say that u satisfies the boundary condition (1.2) if $\eta \circ u \in W^{1,1}_0(\Omega)$ for all $\eta \in C^1(\mathbb{R}^m, \mathbb{R}^m) \cap L^\infty(\mathbb{R}^m, \mathbb{R}^m)$ that satisfy $\eta \equiv id$ on $B(0, r)$ for some $r > 0$ and $|D\eta(y)| \le C(1 + |y|)^{-1}$ for some constant $C < \infty$.

Remarks. 1) The conditions in Definition 1 (except (ii)) are the weakest possible in order to define the equation (1.1) in the sense of distributions. Note that if u is approximately differentiable, then ap Du is measurable and hence $\sigma(\cdot, u, \mathrm{ap}\ Du)$ is measurable.

2) The assumption $\eta \circ u \in W^{1,1}(\Omega; \mathbb{R}^m)$ ensures minimal regularity of u. For example, if $\mu = 0$ and $\sigma(x, u, p) = \sigma(p)$ with $\sigma(0) = 0$, then piecewise constant functions u satisfy ap $Du = 0$ almost everywhere but are not admissible solutions.

3) Note that the class of functions η permitted in the definition of the boundary values includes smooth functions of the form $\eta(y) = \alpha(|y|)\frac{y}{|y|}$ with $\lim_{t\to\infty} \alpha(t) \neq 0$.

The following theorem is the main result in this paper (see the end of this introduction for the definition of the weak Lebesgue space $L^{s,\infty}$).

Theorem 2 *Let Ω be a bounded, open set and suppose that the hypotheses (H0)–(H3) hold. Assume in addition that one of the following conditions is satisfied:*

(i) *$F \mapsto \sigma(x, u, F)$ is a C^1 function.*

(ii) *There exists a function* $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *such that* $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$ *and* $F \mapsto W(x,u,F)$ *is convex and* C^1.

(iii) σ *is strictly monotone, i.e.,* σ *is monotone and* $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) = 0$ *implies* $F = G$.

Let μ *denote an* $\mathbb{R}^m$*-valued Radon measure on* Ω *with finite mass. Then the system (1.1), (1.2) has a solution* u *in the sense of Definition 1 which satisfies the weak Lebesgue space estimate*

$$\|u\|^*_{L^{s^*,\infty}(\Omega)} + \|\mathrm{ap}\, Du\|^*_{L^{s,\infty}(\Omega)} \le C(c_1, c_2, \|\mu\|_{\mathscr{M}}, \mathrm{meas}\,\Omega). \tag{1.3}$$

Here

$$s = \frac{n}{n-1}(p-1)$$

and

$$s^* = \frac{n}{n-p}(p-1)$$

is the Sobolev exponent of s. *If* $c_2 = 0$ *the right hand side of (1.3) reduces to* $C(c_1)\|\mu\|_{\mathscr{M}}^{\frac{1}{p-1}}$.

Remarks. 1) If $p > n$ one can replace the $L^{s^*,\infty}$-norm of u in (1.3) by the $C^{0,\beta}$-norm with $\beta = 1 - \frac{n}{p}$. For $p = q = n$ it is an open question whether $Du \in L^{n,\infty}$. See Section 7 for the (weaker) inclusion $u \in BMO_{\mathrm{loc}}$.

2) The exponents in (1.3) are optimal as can be seen from the nonlinear Green's function $G_p(x) = c|x|^{-n/s^*}$ for the p-Laplace equation

$$-\mathrm{div}(|Du|^{p-2}Du) = \delta_0$$

in $\mathbb{R}^n$, $n \ge 3$. In particular $L^{s,\infty}$ cannot be replaced by L^s.

3) The pointwise monotonicity condition can be replaced by a weaker integrated version, called quasimonotonicity, see Definition 3 and Corollary 4 below.

The key point in the proof of the theorem, which we give in Section 6, is the "div-curl inequality" in Lemma 11 for the Young measure $\{\nu_x\}_{x\in\Omega}$ generated by a sequence Du_k of gradients of approximate solutions. Together with the identity

$$\mathrm{ap}\, Du(x) = \langle \nu_x, \mathrm{Id}\,\rangle \tag{1.4}$$

the div-curl inequality implies easily that $\sigma(\cdot, u_k, Du_k)$ converges weakly in L^1 to $\sigma(\cdot, u, \mathrm{ap}\ Du)$ (see Lemmata 12 and 13 for details). The identity (1.4) is a consequence of general properties of Young measures if $p > 2 - \frac{1}{n}$ since in this case Du_k is bounded in L^s for some $s > 1$. If $1 < p \le 2 - \frac{1}{n}$ one only has the weaker bounds

$$\int_{|u_k|\le\alpha} |Du_k|^p dx \le C(\alpha)$$

but this still suffices to derive (1.4) (see Lemma 9). The main point here, as well as in the proof of the div-curl inequality, is that while ap Du may not be bounded in L^1 it still behaves at almost every point as an L^1 function (and even as a C^0 function up to a set of density zero). Young measures achieve a sufficient localization to exploit that fact.

We will also use a weaker, integrated version of the pointwise definition of monotonicity (H1) which we call quasimonotonicity. The definition is phrased in terms of gradient Young measures (see Section 2 for further details). Note, however, that although quasimonotonicity is "monotonicity in integrated form", the gradient $D\eta$ of a quasiconvex function η is not necessarily quasimonotone.

Definition 3 *A function $\eta\colon \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is said to be strictly p-quasimonotone if*

$$\int_{\mathbb{M}^{m\times n}} (\eta(\lambda) - \eta(\overline{\lambda})) : (\lambda - \overline{\lambda})\, d\nu(\lambda) > 0$$

for all homogeneous $W^{1,p}$-gradient Young measures ν with centre of mass $\overline{\lambda} = \langle \nu, \mathrm{Id}\rangle$ which are not a single Dirac mass.

A simple example is the following: Assume that η satisfies the growth condition

$$|\eta(F)| \le C\, |F|^{p-1}$$

with $p > 1$ and the structure condition

$$\int_{\Omega} (\eta(F + \nabla\varphi) - \eta(F)) : \nabla\varphi\, dx \ge c \int_{\Omega} |\nabla\varphi|^p\, dx$$

for all $\varphi \in C_0^\infty(\Omega)$ and all $F \in \mathbb{M}^{m\times n}$. Then η is strictly p-quasimonotone. This follows easily from the definition if one uses that for every $W^{1,p}$-gradient Young measure ν there exists a sequence $\{Dv_k\}$ generating ν for which $\{|Dv_k|^p\}$ is equiintegrable (see [FMP], [KP]).

As a consequence of our results we state the following corollary:

Corollary 4 *Assume that the hypotheses (H0), (H2), (H3) are satisfied and that σ is strictly p-quasimonotone. Let μ be an $\mathbb{R}^m$-valued Radon measure on Ω with finite mass. Then the system (1.1), (1.2) has a solution in the sense of Definition 1 and the a priori estimate (1.3) holds.*

Our results generalize recent results in [FR] and [DHM] for the p-Laplace system. The main improvements with respect to existing results are the relatively weak assumptions in Theorem 2 and Corollary 4. In particular it suffices to assume monotonicity or the weaker p-quasimonotonicity condition instead of strict monotonicity. Moreover different coercivity and growth rates are allowed and the case $p \le 2 - \frac{1}{n}$ is included. For another approach to such questions see [DMM1] and [DMM2].

There exists an extensive literature on elliptic and parabolic equations with measure valued right hand side see, e.g., [BB], [BG], [BM], [LM], [Mu1], [Mu2], [Mu3], [Ra] and the literature cited therein. Compactness questions have been discussed in [Fr], [La], [Zh]. Partial results concerning uniqueness of solutions can be found in [BB], [DA], [KX], [LM].

We end this introduction by recalling the definition of the weak Lebesgue spaces $L^{s,\infty}$. A measurable function $f : \Omega \to \mathbb{R}^l$ belongs to $L^{s,\infty}(\Omega)$ if $\|f\|^*_{L^{s,\infty}} := \sup_{t>0} t^{1/s} f^*(t) < \infty$ where $f^*(t) := \inf\{y > 0 : \lambda_f(y) \le t\}$ is the non-increasing rearrangement of f and $\lambda_f(y) = \mathscr{L}^n\{|f| > y\}$ is the distribution function of f. The expression $\|f\|^*_{L^{s,\infty}}$ is only a quasinorm, but for $s > 1$ it is equivalent to the usual norm of $L^{s,\infty}$. For more information about topological properties of the Lorentz spaces $L^{s,r}$ (in particular for $0 < s \le 1$) see [Hu].

2 A brief review of Young measures

In this section we briefly summarize basic facts concerning Young measures. We follow the formulation given by Ball (see [Bl] and references therein). The fundamental theorem about Young measures may be stated as follows:

Theorem 5 (Young, Tartar, Ball) *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $z_j : \Omega \to \mathbb{R}^m$, $j = 1, 2, \ldots$, be a sequence of Lebesgue measurable functions. Then there exists a subsequence z_k and a family $\{\nu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that*

(i) $\|\nu_x\| := \int d\nu_x \le 1$ *for almost every* $x \in \Omega$

(ii) $\varphi(z_k) \overset{*}{\rightharpoonup} \bar{\varphi}$ *weakly* in* $L^\infty(\Omega)$ *for all* $\varphi \in C^0_0(\mathbb{R}^n)$, *where* $\bar{\varphi}(x) = \langle \nu_x, \varphi\rangle$

(iii) If for all $R > 0$

$$\lim_{L\to\infty} \sup_{k\in\mathbb{N}} \operatorname{meas}\{x \in \Omega \cap B(0,R) : |z_k(x)| \ge L\} = 0 \tag{2.1}$$

then $\|\nu_x\| = 1$ *for almost every* $x \in \Omega$, *and for all measurable* $A \subset \Omega$ *there holds* $\varphi(z_k) \rightharpoonup \bar{\varphi} = \langle \nu_x, \varphi\rangle$ *weakly in* $L^1(A)$ *for continuous* φ *provided the sequence* $\varphi(z_k)$ *is weakly precompact in* $L^1(A)$.

Here, "meas" denotes the Lebesgue measure restricted to Ω and $C^0_0(\mathbb{R}^m) = \{\varphi \in C^0(\mathbb{R}^m) : \lim_{|z|\to\infty} |\varphi(z)| = 0\}$.

Notice, that under hypothesis (2.1) for any measurable $A \subset \Omega$,

$$\varphi(\cdot, z_k) \rightharpoonup \langle \nu_x, \varphi(x,\cdot)\rangle \quad \text{weakly in } L^1(A) \tag{2.2}$$

for every Carathéodory function $\varphi\colon A \times \mathbb{R}^m \to \mathbb{R}$ provided the sequence $\{\varphi(\cdot, z_k)\}$ is weakly precompact in $L^1(A)$ (see [Bl]). Moreover, if $\mathscr{L}^n(\Omega) < \infty$,

$$z_k \to z \text{ in measure} \iff \text{the Young measure associated to } z_k \text{ is } \delta_{z(x)}. \tag{2.3}$$

The Young measure associated to the sequence (y_k, z_k) is

$$\delta_{y(x)} \otimes \nu_x \tag{2.4}$$

if $y_k \to y$ in measure and if ν_x is the Young measure associated to z_k.

A Young measure $\{\nu_x\}_{x\in\Omega}$ is called $W^{1,p}$-gradient Young measure ($1 \le p \le \infty$) if it is associated to a sequence of gradients $\{Du_k\}$ such that $\{u_k\}$ is bounded in $W^{1,p}(\Omega)$. It is called homogeneous if $\nu_x = \mu$ for almost every

$x \in \Omega$. If $\{\nu_x\}_{x\in\Omega}$ is a $W^{1,p}$-gradient Young measure then there exists a function $u \in W^{1,p}(\Omega)$ such that $Du(x) = \langle \nu_x, \mathrm{Id}\,\rangle$ almost everywhere.

The following Fatou-type lemma will be useful in Section 5:

Lemma 6 *Let $F\colon \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k\colon \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that $u_k \to u$ in measure and such that Du_k generates the Young measure ν_x. Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k(x), Du_k(x))\, dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \lambda)\, d\nu_x(\lambda)\, dx \tag{2.5}$$

provided that the negative part $F^-(x, u_k(x), Du_k(x))$ is equiintegrable.

More general versions of this lemma may be found in [Bd1], [Bd2] and [Val1], [Val2]. Our assumptions allow the following elementary proof.

Proof. We may assume that the limes inferior on the left-hand side of (2.5) agrees with the limit and is finite. Consider the Carathéodory functions $F_R(x,u,p) = \min\{R, F(x,u,p)\}$ for $R > 0$. For fixed $R > 0$ the sequence $\{F_R(x,u_k(x), Du_k(x))\}_k$ is equiintegrable. We have

$$\int_\Omega F_R(x, u_k(x), Du_k(x))\, dx \leq \int_\Omega F(x, u_k(x), Du_k(x))\, dx \leq C < \infty$$

for all k and $R > 0$. By (2.2) we have that for all $R > 0$

$$\lim_{k\to\infty} \int_\Omega F_R(x, u_k(x), Du_k(x))\, dx = \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx \leq C,$$

and by monotone convergence of the integrands as $R \to \infty$

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx \leq C < \infty\,. \tag{2.6}$$

On the other hand

$$\begin{aligned}
&\int_\Omega F(x, u_k(x), Du_k(x))\, dx - \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx = \\
&= \quad \int_\Omega F(x, u_k(x), Du_k(x))\, dx - \int_\Omega F_R(x, u_k(x), Du_k(x))\, dx + \\
&\qquad + \int_\Omega F_R(x, u_k(x), Du_k(x))\, dx - \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx + \\
&\qquad + \int_\Omega \int_{\mathbb{M}^{m\times n}} F_R(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx - \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u(x), \lambda)\, d\nu_x(\lambda)\, dx \\
&=: \quad I_k + II_k + III\,.
\end{aligned}$$

Now we have

$$\begin{aligned} &I_k \geq 0\,,\\ &II_k \to 0 \quad \text{for any fixed } R>0 \text{ as } k\to\infty,\\ &III \to 0 \quad \text{as } R\to\infty, \text{ because of (2.6) and monotone convergence,} \end{aligned}$$

and the claim follows. □

3 Refined convergence results for $1 < p \leq 2 - \frac{1}{n}$

We shall see in the next section that solutions $u_k \in W_0^{1,p}(\Omega)$ of the system

$$-\mathrm{div}\, \sigma(x, u_k(x), Du_k(x)) = f_k$$

with $f_k \in C^\infty(\Omega)$ satisfy the a priori estimate

$$\int_{|u_k|\leq\alpha} |Du_k|^p\, dx \leq C(\alpha, \|f_k\|_{L^1}). \tag{3.1}$$

If $p > 2 - \frac{1}{n}$ one can deduce (see Lemma 10 below) that Du_k is uniformly bounded in some $L^s(\Omega)$ with $s > 1$. For $p \leq 2 - \frac{1}{n}$, however, Du_k may not be bounded in L^1 and hence it is not clear in what sense Du_k converges and whether the (weak) limit of u_k is differentiable in any sense. This difficulty has in fact led to the restriction $p > 2 - \frac{1}{n}$ in many previous results.

In this section we show how Young measures can be used to extract from (3.1) almost the same information as from uniform L^p estimates of the gradient. In particular we show that, for $p > 1$, the estimate (3.1) implies pointwise almost everywhere convergence of (a subsequence of) u_k (see Lemma 8) and approximate differentiability of the limit as well as the important identity

$$\mathrm{ap}\, Du(x) = \langle \nu_x, \mathrm{Id}\,\rangle \quad \text{almost everywhere in } \Omega$$

(see Lemma 9). In the following T_α denotes the truncation function

$$T_\alpha(y) = \min\{1, \frac{\alpha}{|y|}\}y\,, \quad \alpha > 0.$$

By definition $|T_\alpha(y)| \leq \alpha$ and

$$DT_\alpha(y) = \begin{cases} \mathrm{Id} & \text{for } |y| < \alpha,\\ \frac{\alpha}{|y|}(\mathrm{Id} - \frac{y}{|y|} \otimes \frac{y}{|y|}) & \text{for } |y| > \alpha. \end{cases}$$

Lemma 7 *Let $u_k : \Omega \to \mathbb{R}^m$ be a sequence of measurable functions such that*

$$\sup_{k\in\mathbb{N}} \int_\Omega |u_k|^s\, dx < \infty \quad \textit{for some } s > 0. \tag{3.2}$$

Suppose that for each $\alpha > 0$ the sequence of truncated functions

$$\{T_\alpha(u_k)\}_{k\in\mathbb{N}} \quad \textit{is precompact in } L^1(\Omega).$$

Then there exists a measurable function u on Ω such that for a subsequence

$$u_{k_j} \to u \quad \textit{in measure.}$$

Proof. Choose a subsequence of $\{u_k\}$ (not relabeled) which generates a Young measure $\{\nu_x\}_{x\in\Omega}$. By (3.2) and Theorem 5(iii) the measures ν_x are probability measures for almost every $x \in \Omega$ and

$$T_\alpha(u_k) \rightharpoonup v_\alpha = \langle \nu_x, T_\alpha \rangle$$

weakly in $L^1(\Omega, \mathbb{R}^m)$ and in fact strongly since $T_\alpha(u_k)$ is precompact in L^1. Consequently there exists a subsequence such that

$$T_\alpha(u_{k_l}) \to v_\alpha \quad \text{almost uniformly,} \tag{3.3}$$

i.e., $T_\alpha(u_{k_l}) \to v_\alpha$ uniformly up to a set of arbitrary small measure. Let

$$M_\alpha = \{x \in \Omega \,:\, |v_\alpha(x)| < \alpha\}\,.$$

Then for each $\varepsilon > 0$ and $\delta > 0$ there exists a set E_ε of measure $\operatorname{meas}(E_\varepsilon) < \varepsilon$ and an index $l_0(\varepsilon, \delta)$ such that

$$|T_\alpha(u_{k_l})| < |v_\alpha(x)| + \delta \quad \text{for all } x \in M_\alpha \setminus E_\varepsilon \text{ and all } l \geq l_0\,.$$

It follows that

$$u_{k_l}(x) \to v_\alpha(x) \quad \text{for almost every } x \in M_\alpha \setminus E_\varepsilon$$

(consider first $x \in M_\beta$, $\beta < \alpha$ and then the union over $\beta < \alpha$). Since $\varepsilon > 0$ was arbitrary it follows that

$$\nu_x = \delta_{v_\alpha(x)} \quad \text{for almost every } x \in M_\alpha.$$

In view of the equivalence (2.3) it suffices to show that $\cup M_\alpha$ has full measure. Now clearly $M_\alpha \subset M_\beta$ for $\alpha < \beta$ since

$$T_\beta(u_{k_l}) \to T_\beta(v_\alpha) = v_\alpha \quad \text{almost everywhere in } M_\alpha,$$

and therefore $v_\beta = v_\alpha$ on M_α. By (3.3) there exists for each $\varepsilon > 0$ a set E_ε, and an index $l_0(\varepsilon, \alpha)$ such that $\operatorname{meas}(E_\varepsilon) < \varepsilon$ and

$$|u_{k_l}| \geq |T_\alpha(u_{k_l})| \geq \frac{\alpha}{2} \quad \text{on } (\Omega \setminus E_\varepsilon) \setminus M_\alpha \text{ for all } l \geq l_0.$$

In view of (3.2) this implies

$$\operatorname{meas}((\Omega \setminus E_\varepsilon) \setminus M_\alpha) \leq \frac{C}{\alpha^s}\,.$$

Letting $\varepsilon \to 0$ we deduce

$$\operatorname{meas}(\Omega \setminus \cup M_\alpha) = \lim_{\alpha\to\infty} \operatorname{meas}(\Omega \setminus M_\alpha) = 0$$

and the proof is finished. □

Lemma 8 *Let Ω be a domain in $\mathbb{R}^n$ with $\mathscr{L}^n(\Omega) < \infty$ and $u_k \in W^{1,1}(\Omega; \mathbb{R}^m)$. Suppose that there exist $p \geq 1$ and $s > 0$ such that*

$$\sup_{k\in\mathbb{N}} \int_{|u_k|\leq\alpha} |Du_k|^p dx \leq C(\alpha) < \infty \quad \text{for all } \alpha > 0 \tag{3.4}$$

and

$$\sup_{k\in\mathbb{N}} \int_{\Omega} |u_k|^s dx \leq C < \infty\,.$$

Then there exist a subsequence u_{k_j} and a measurable function $u\colon \Omega \to \mathbb{R}^m$ such that

$$u_{k_j} \to u \quad \text{in measure.}$$

Moreover u is for almost every $x \in \Omega$ approximately differentiable. For all $\eta \in C_0^\infty(\mathbb{R}^m; \mathbb{R}^m)$ there holds $\eta \circ u \in W^{1,p}(\Omega; \mathbb{R}^m)$. If $u_k \in W_0^{1,1}(\Omega)$ then $\eta \circ u \in W_0^{1,1}(\Omega) \cap W^{1,p}(\Omega)$ provided that $\eta \equiv id$ on $B(0, r)$ for some $r > 0$.

Remark. If $C(\alpha) \leq C'(\alpha + 1)$ and $p > 1$ then the assertion holds for all $\eta \in C^1(\mathbb{R}^m; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^m; \mathbb{R}^m)$ that satisfy $|D\eta(y)| \leq C(1 + |y|)^{-1}$. To see this, it suffices to verify that $D(\eta \circ u_k)$ is bounded in $L^p(\Omega)$. This follows by an application of (3.4) with $\alpha = 2^j, j \in \mathbb{N}$.

Proof. Choose a subsequence (not relabeled) of the sequence $\{u_k\}$ which generates a Young measure $\{\nu_x\}_{x\in\Omega}$. Suppose first in addition that Ω is such that the compact Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ holds. Note that by (3.4)

$$\|D(T_\alpha(|u_k|))\|_{L^p(\Omega)}^p \leq C(\alpha)\,.$$

Hence by the compact Sobolev embedding the sequence $\{T_\alpha(|u_k|)\}$ is precompact in L^1 and by Lemma 7 there exists a measurable function w such that (after passage to a subsequence)

$$|u_k| \to w$$

in measure. It follows that

$$\operatorname{spt} \nu_x \subset S_{w(x)} = \{y \in \mathbb{R}^m : |y| = w(x)\}. \tag{3.5}$$

Let $M_\alpha = \{x \in \Omega\colon |w(x)| < \alpha\}$ and choose a radially symmetric cut-off function $\eta \in C_0^\infty(B(0, 3\alpha); \mathbb{R}^m)$ such that $\eta \equiv \mathrm{Id}$ on $B(0, 2\alpha)$. Then by (3.4) and by the compact Sobolev embedding $\eta(u_k)$ is precompact in $L^p(\Omega)$ and thus

$$\eta(u_k) \to v \quad \text{in measure.}$$

Hence

$$\operatorname{spt} \nu_x \subset \eta^{-1}(v(x))\,. \tag{3.6}$$

If $v(x) \neq 0$ then $\eta^{-1}(v(x))$ is concentrated on the ray through $v(x)$ and it follows from (3.5) and (3.6) that ν_x is a Dirac mass. If $v(x) = 0$ then $\eta^{-1}(v(x)) \subset$

$\{0\} \cup (\mathbb{R}^n \setminus B(0, 2\alpha))$. For $x \in M_\alpha$ one deduces from (3.5) and (3.6) that $\nu_x = \delta_0$. Hence ν_x is a Dirac mass for almost every $x \in M_\alpha$ and thus for almost every $x \in \Omega$ since $\cup_{\alpha>0} M_\alpha = \Omega \setminus E$ where E is a set of measure zero. Therefore u_k converges by (2.3) in measure to a measurable function u.

Now we remove the additional regularity restriction on Ω. Let $\Omega_k \subset \Omega$ be a sequence of Lipschitz domains (choose, e.g., a finite union of balls for Ω_k) such that $\mathscr{L}^n(\Omega \setminus \Omega_k) \to 0$ as $k \to \infty$. Application of the previous arguments to Ω_k shows that ν_x is a Dirac mass for almost every $x \in \Omega_k$. Hence ν_x is a Dirac mass for almost every $x \in \Omega$ and $u_k \to u$ in measure, where $u(x) := \langle \nu_x, \mathrm{Id} \rangle$.

To see that u is approximately differentiable, let $M_\alpha = \{x \in \Omega \colon |u(x)| < \alpha\}$. It suffices to show that u is almost everywhere approximately differentiable in M_α for all $\alpha > 0$. For η as above we have

$$\eta(u_k) \rightharpoonup \eta(u) \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^m).$$

In particular, $\eta(u)$ is almost everywhere approximately differentiable. Let $x_0 \in M_\alpha$ be a point of approximate differentiability of $\eta(u)$ and of approximate continuity of u, i.e.,

$$\lim_{r\to 0} \frac{1}{r^n} \text{meas}\, \{x \in B(x_0, r) : |u(x) - u(x_0)| > \delta\} = 0, \qquad \text{for all } \delta > 0.$$

For $\varepsilon > 0$ consider the set

$$E_{r,\varepsilon} = \{x \in B(x_0, r) : |u(x) - u(x_0) - \text{ap}\, D(\eta \circ u)(x_0)(x - x_0)| > \varepsilon r\}.$$

Then, by the approximate continuity of u,

$$\lim_{r\to 0} \frac{1}{r^n} \text{meas}(E_{r,\varepsilon} \cap \{|u(x) - u(x_0)| \geq \frac{\alpha}{2}\}) = 0,$$

while

$$\lim_{r\to 0} \frac{1}{r^n} \text{meas}(E_{r,\varepsilon} \cap \{|u(x) - u(x_0)| < \frac{\alpha}{2}\}) = 0,$$

since u and $\eta \circ u$ agree on that set and $\eta \circ u$ is approximately differentiable at x_0. Hence u is approximately differentiable at x_0 and $\text{ap}\, Du = \text{ap}\, D(\eta \circ u)(x_0)$. □

Lemma 9 *Let u_k be as in Lemma 8 with $p > 1$. Then the Young measure ν_x generated by (a subsequence of) Du_k has the following properties:*

(a) ν_x is a probability measure for almost every $x \in \Omega$.
(b) ν_x has finite p-th moment for almost every $x \in \Omega$, i.e., $\int_{\mathbb{M}^{m\times n}} |\lambda|^p d\nu_x(\lambda)$ is finite for almost every $x \in \Omega$.
(c) ν_x satisfies

$$\langle \nu_x, \mathrm{Id} \rangle = \text{ap}\, Du(x) \quad \textit{almost everywhere in } \Omega.$$

(d) ν_x is a homogeneous $W^{1,p}$-gradient Young measure for almost every $x \in \Omega$.

Proof. Let $\tilde{\nu}_x$ denote the Young measure generated by (a subsequence of) the sequence $\{(u_k, Du_k)\}$. By Lemma 8 we have

$$\tilde{\nu}_x = \delta_{u(x)} \otimes \nu_x \,.$$

Let $\eta \in C_0^\infty(B(0,2\alpha);\mathbb{R}^m)$, $\eta \equiv \mathrm{Id}$ on $B(0,\alpha)$, and let ν^η be the Young measure generated by

$$D(\eta \circ u_k) = (D\eta)(u_k)\, Du_k \,.$$

Then ν_x^η is a probability measure, has finite p-th moment and

$$\langle \nu_x^\eta, \mathrm{Id}\, \rangle = (D(\eta \circ u))(x) = D\eta(u(x))\,\mathrm{ap}\; Du(x)\,.$$

It follows for $\varphi \in C_0^\infty(\mathbb{M}^{m\times n})$ that

$$\varphi(D(\eta \circ u_k)) \rightharpoonup \langle \nu_x^\eta, \varphi \rangle = \int_{\mathbb{M}^{m\times n}} \varphi(\lambda)\, d\nu_x^\eta(\lambda)\,.$$

Rewriting the left hand side we have on the other hand

$$\begin{aligned} \varphi((D\eta)(u_k)\, Du_k) \quad &\rightharpoonup \quad \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} \varphi(D\eta(\rho)\lambda)\, d\tilde{\nu}_x(\rho,\lambda) \\ &= \int_{\mathbb{M}^{m\times n}} \varphi(D\eta(u(x))\lambda)\, d\nu_x(\lambda)\,. \end{aligned}$$

Hence

$$\nu_x^\eta = \nu_x \quad \text{if } |u(x)| < \alpha. \tag{3.7}$$

Therefore the properties in (a), (b) and (c) hold for almost every $x \in \{|u| < \alpha\}$ since they hold for ν_x^η. Taking the union over $\alpha > 0$ we obtain (a), (b) and (c).

To prove (d) note that

$$\int_\Omega |D(\eta \circ u_k)|^p\, dx \le \sup |D\eta|^p \int_{|u_k| \le 2\alpha} |Du_k|^p\, dx \le \sup |D\eta|^p\, C(2\alpha)\,.$$

By the localization principle in [KP] we conclude that ν_x^η is a homogeneous $W^{1,p}$-gradient Young measure for almost every $x \in \Omega$. Thus (d) follows from (3.7) and the fact that α was arbitrary. □

4 Approximate solutions and a priori bounds

Throughout this section we assume that $p = q$ in (H2), i.e., that the growth and coercivity rate of σ coincide. In order to establish existence of a solution of (1.1), (1.2) we introduce the following approximating problems:

$$-\mathrm{div}\; \sigma(x, u_k(x), Du_k(x)) = f_k(x) \qquad \text{in } \Omega, \tag{4.1}$$

$$u_k = 0 \qquad \text{on } \partial\Omega. \tag{4.2}$$

For f_k we choose the standard mollification

$$f_k(x) = \int_{\mathbb{R}^n} \gamma_k(x-y)\, d\mu(y)$$

where, for $k \in \mathbb{N}$, $\gamma_k(x) = k^n \gamma_0(kx)$ with a function $\gamma_0 \in C_0^\infty(B(0,1))$, $\gamma_0 \geq 0$, $\|\gamma_0\|_{L^1} = 1$. Then $f_k \in C^\infty \cap L^1 \cap L^\infty$ for each k and

$$f_k \overset{*}{\rightharpoonup} \mu \qquad \text{in } \mathscr{M}.$$

Let $A\colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ denote the operator

$$A\colon u \mapsto \left(v \mapsto \int_\Omega \sigma(x, u(x), Du(x)) : Dv\, dx \right).$$

By (H0) and (H2) this operator is well defined. If we assume for simplicity (see also the remark below) that σ only depends on x and Du but not on u then, by (H1), the operator is monotone, i.e.,

$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad \text{for all } u, v \in W_0^{1,p}(\Omega)$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of $W_0^{1,p}$ and $W^{-1,p'}$. The coercivity hypothesis in (H2) implies that A is coercive, i.e., $\langle A(u), u \rangle \geq c(\|u\|_{W^{1,p}})\|u\|_{W^{1,p}}$ for a real valued function c with $\lim_{t\to\infty} c(t) = \infty$. On the other hand the growth condition in (H2) (with $q = p$) implies that A is hemicontinuous, i.e., the mapping $t \mapsto \langle A(u + tv), w \rangle$ is continuous on the real axis for $u, v, w \in W_0^{1,p}(\Omega)$. Then by a standard theorem for monotone operators (see, e.g., [Va]) it follows that A is surjective and hence that (4.1), (4.2) has a solution $u_k \in W_0^{1,p}(\Omega)$ for all $k \in \mathbb{N}$.

Remark. If σ depends explicitly on u or if σ is merely strictly p-quasimonotone rather than monotone it is slightly more difficult to show existence of solutions of (4.1), (4.2). However, using Borsuk's theorem one can solve (4.1), (4.2) approximately in finite dimensional subspaces of $W_0^{1,p}(\Omega)$ and then pass to the limit by a suitable adaptation of Lemma 13 below.

As in [DHM], one easily derives a priori estimates for the solutions u_k of the approximating problems.

Lemma 10 *Let $\Omega \subset \mathbb{R}^n$ be an open set, $f \in L^1(\Omega; \mathbb{R}^m)$. Assume that σ satisfies (H2) and (H3) with $p = q$ and that $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$ is a solution of*

$$-\operatorname{div} \sigma(x, u(x), Du(x)) = f \tag{4.3}$$

in the sense of distributions. Then

$$\begin{aligned} u &\in L^{s^*,\infty}(\Omega; \mathbb{R}^m), \\ Du &\in L^{s,\infty}(\Omega; \mathbb{M}^{m\times n}) \end{aligned}$$

where

$$s = \frac{n}{n-1}(p-1), \quad s^* = \frac{ns}{n-s} = \frac{n}{n-p}(p-1)$$

and

$$\|Du\|^*_{L^{s,\infty}} + \|u\|^*_{L^{s^*,\infty}} \le C(c_1, c_2, \|f\|_{L^1}, \text{meas}\,\Omega). \tag{4.4}$$

If $c_2 = 0$ *the right hand side in (4.4) reduces to* $C(c_1)\|f\|_{L^1}^{\frac{1}{p-1}}$.

Proof. We use similar techniques as Talenti [Ta] in connection with quasilinear elliptic equations and also as Bénilan et al. [BB, Lemma 4.1] for solutions of the p-Laplace equation. As above define the truncation function T_α by $T_\alpha(y) = \min(1, \frac{\alpha}{|y|})y$. By definition $|T_\alpha(y)| \le \alpha$ and

$$DT_\alpha(y) = \begin{cases} \text{Id} & \text{for } |y| < \alpha, \\ \frac{\alpha}{|y|}(\text{Id} - \frac{y}{|y|} \otimes \frac{y}{|y|}) & \text{for } |y| > \alpha. \end{cases}$$

Testing (4.3) with $T_\alpha(u)$ and observing (H2) and (H3), we obtain

$$c_1 \int_{|u|<\alpha} |Du|^p\, dx \le \alpha\|f\|_{L^1(\Omega)} + c_2\text{meas}(\Omega)\,. \tag{4.5}$$

Using the fact that $|Du| \ge |D|u||$ and defining $u_\alpha = \min(|u|, \alpha)$, we obtain from the Sobolev embedding theorem that

$$c\int_\Omega |u_\alpha|^{p^*} dx \le (\alpha\|f\|_{L^1(\Omega)} + c_2\text{meas}(\Omega))^{p^*/p}.$$

Hence we may estimate the distribution function $\lambda_{|u|}$ of $|u|$ by

$$\begin{aligned} \lambda_{|u|}(\alpha) &\le \alpha^{-p^*}\int_\Omega |u_\alpha|^{p^*} dx \\ &\le c\max(\alpha^{-s^*}\|f\|^{p^*/p}_{L^1(\Omega)}, (c_2\text{meas}\,\Omega)^{p^*/p}\alpha^{-p^*}) \end{aligned}$$

and trivially

$$\lambda_{|u|}(\alpha) \le \text{meas}\,\Omega.$$

The combination of these two estimates implies

$$\|u\|^*_{L^{s^*,\infty}} \le C\ \max(\|f\|_{L^1}^{\frac{1}{p-1}}, c_2^{\frac{1}{p}}(\text{meas}\,\Omega)^{\frac{1}{s}}). \tag{4.6}$$

From (4.5) and (4.6) one deduces that the distribution function $\lambda_{|Du|}$ satisfies for all $\alpha > 0$

$$\begin{aligned} \lambda_{|Du|}(\beta) &\le \frac{1}{\beta^p}\int_{|u|<\alpha} |Du|^p\, dx + \lambda_{|u|}(\alpha) \\ &\le \frac{c}{\beta^p}(\alpha\|f\|_{L^1} + c_2\text{meas}\,\Omega) + \\ &\quad + c\max(\alpha^{-s^*}\|f\|^{p^*/p}_{L^1(\Omega)}, (c_2\text{meas}\,\Omega)^{p^*/p}\alpha^{-p^*}). \end{aligned}$$

Choosing $\alpha = \beta^{\frac{n-p}{n-1}}$ and observing that $\lambda_{|Du|} \le \text{meas}\,\Omega$ we deduce (4.4). Repeating the proof with $c_2 = 0$ we easily establish the form of the constant in that particular case. □

5 A div-curl inequality

The result of this section is the key ingredient for the proof that one can pass to the limit in the equation (4.1) for the solutions $\{u_k\}_{k\in\mathbb{N}}$ of approximating problems. Since it is independent of the differential equation we state it in a more general form using only the hypotheses (5.1)–(5.7) below. Using Lemmata 8 and 10 it is easily verified that they hold under the assumptions in Section 4.

(5.1) $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function.
(5.2) $\sigma(x,u,F) : MF \geq 0$ holds for all matrices $M = \mathrm{Id} - b\otimes b \in \mathbb{M}^{m\times m}$ with $|b| \leq 1$.
(5.3) $u_k \in W^{1,1}(\Omega;\mathbb{R}^m)$ and there exists an $s > 0$ such that $\int_\Omega |Du_k|^s dx \leq C$ uniformly in k.
(5.4) The sequence $\sigma_k(x) = \sigma(x, u_k(x), Du_k(x))$ is equiintegrable.
(5.5) The sequence u_k converges in measure to some function u, and u is almost everywhere approximately differentiable.
(5.6) The sequence $f_k := -\mathrm{div}\,\sigma_k$ is bounded in $L^1(\Omega)$.
(5.7) $Du_k \in L^r_{\mathrm{loc}}$ and $\sigma_k \in L^{r'}_{\mathrm{loc}}$ for some r, $1 < r < \infty$.

Remark. Assumption (5.2) coincides with condition (A5) in [La] if $\sigma(x,u,F) : F \geq 0$. This condition could be relaxed to $\sigma(x,u,p) : Mp \geq -C|p|^\beta$ if the sequence $|Du_k|^\beta$ is equiintegrable.

We may assume (after passing to a suitable subsequence if necessary) that $\{Du_k\}$ generates a Young measure ν. It follows from Theorem 5(iii) and (5.3) that ν_x is a probability measure for almost every $x \in \Omega$.

Lemma 11 *Suppose (5.1)–(5.7). Then (after passage to a subsequence) the sequence σ_k converges weakly in $L^1(\Omega)$ and the weak limit $\bar\sigma$ is given by $\bar\sigma(x) = \langle \nu_x, \sigma(x,u(x),\cdot)\rangle$. Moreover the following inequality holds:*

$$\int_{\mathbb{M}^{m\times n}} \sigma(x,u(x),\lambda) : \lambda \, d\nu_x(\lambda) \leq \bar\sigma(x) : \mathrm{ap}\, Du(x) \quad \textit{for a.e. } x \in \Omega.$$

Remarks. 1) The assertion of Lemma 11 follows (with equality) directly from the div-curl lemma (see [Mu1], [Tar]) if $f_k = 0$, if $\{u_k\}$ is bounded in $W^{1,p}(\Omega)$ and if $\{\sigma_k\}$ is bounded in $L^{p'}(\Omega)$ with $1 < p < \infty$.

2) If the sequence $\{Du_k\}$ is equiintegrable then, by Theorem 5(iii), $Du(x) = \mathrm{ap}\, Du(x) = \langle \nu_x, \mathrm{Id}\rangle$ almost everywhere.

Proof. Choose a non-negative function $\alpha_1 \in C^\infty([0,\infty)) \cap L^\infty([0,\infty))$ such that $\alpha_1 = \mathrm{Id}$ on $[0,\delta)$ for some $\delta > 0$, $\alpha_1' \geq 0$ and

$$\alpha_1'(s)s \leq \alpha_1(s) \quad \text{for } s \geq 0. \tag{5.8}$$

One possible choice is $\alpha_1(s) = s$ for $0 \leq s \leq \delta$ and

$$\alpha_1(s) = \delta \exp\Big(\int_\delta^s \frac{\varepsilon(\xi)}{\xi}\, d\xi\Big) \qquad \text{for } s > \delta$$

where the C^∞-function $\varepsilon \in L^1([\,\delta,\infty);[\,0,1\,])$ satisfies $\varepsilon(\delta) = 1$ and $\varepsilon^{(n)}(\delta) = 0$ for all $n \geq 1$. Then let

$$\psi_1(z) = \alpha_1(|z|)\,\frac{z}{|z|} \quad \text{for } z \in \mathbb{R}^m, \tag{5.9}$$

and choose $\varphi_1 \in C_0^\infty(\Omega;\mathbb{R})$ with $\varphi_1 \geq 0$ and $\int_{\mathbb{R}^n} \varphi_1\,dx = 1$. The idea is to multiply the equation in (5.6) by $\varphi_1\,\psi_1 \circ (u_k - v)$ where $v \in C^1(\Omega;\mathbb{R}^m)$ is a suitable comparison function and to use φ_1 to localize the resulting equation

$$\int_\Omega \sigma_k : D(\varphi_1\,\psi_1(u_k - v))dx = \int_\Omega f_k\varphi_1\,\psi_1(u_k - v)dx \tag{5.10}$$

in x. We first estimate the left hand side in (5.10). Let

$$\begin{aligned} h_k &:= \sigma_k : D(\varphi_1\,\psi_1(u_k - v)) \\ &= \sigma_k : \psi_1(u_k - v) \otimes D\varphi_1 + \sigma_k : D\psi_1 \circ (u_k - v)\,D(u_k - v)\,\varphi_1 \end{aligned}$$

and let $\{\mu_x\}_{x\in\Omega}$ be the Young measure generated by the sequence $\{u_k, Du_k\}$. Then, by (5.5) and (2.4),

$$\mu_x = \delta_{u(x)} \otimes \nu_x$$

and thus by (5.4) and (2.2)

$$\sigma_k \rightharpoonup \overline{\sigma} \quad \text{weakly in } L^1(\Omega)$$

with

$$\overline{\sigma}(x) = \int_{\mathbb{R}^m \times \mathbb{M}^{m\times n}} \sigma(x,\rho)d\mu_x(\rho) = \int_{\mathbb{M}^{m\times n}} \sigma(x,u(x),\lambda)d\nu_x(\lambda)\,. \tag{5.11}$$

Note that

$$\begin{aligned} D\psi_1(z) &= (\mathrm{Id} - \theta \otimes \theta)\,\frac{\alpha_1(|z|)}{|z|} + \theta \otimes \theta\,\alpha_1'(|z|) \\ &= \frac{\alpha_1(|z|)}{|z|}(\mathrm{Id} - (1 - \frac{\alpha_1'(|z|)|z|}{\alpha_1(|z|)})\theta \otimes \theta), \end{aligned}$$

where $\theta = \frac{z}{|z|}$. By (5.2) and (5.8) we have

$$\sigma(x,u_k,Du_k) : D\psi_1 \circ (u_k - v)\,Du_k \geq 0 \tag{5.12}$$

and therefore we conclude that the sequence $(h_k)^-$ is equiintegrable. By Lemma 6 and (5.5) we deduce

$$\begin{aligned} \liminf_{k\to\infty} \int_\Omega h_k\,dx \geq& \int_\Omega \overline{\sigma} : \psi_1(u - v) \otimes D\varphi_1\,dx \\ &+ \int_\Omega \varphi_1 \int_{\mathbb{M}^{m\times n}} \sigma(x,u(x),\lambda) : D\psi_1 \circ (u - v)(\lambda - Dv(x))d\nu_x(\lambda)dx. \end{aligned} \tag{5.13}$$

To obtain the first term on the right hand side we used the fact that for two sequences f_k and g_k with $f_k \rightharpoonup f$ weakly in L^r, $r \geq 1$, and $g_k \to g$ boundedly almost everywhere, the product $f_k\, g_k$ converges weakly to $f\, g$ in L^r which is easily verified using Egoroff's theorem and the Lebesgue dominated convergence theorem. To estimate the right hand side in (5.10) note that by (5.6) (after passage to a subsequence if necessary)

$$|f_k| \overset{*}{\rightharpoonup} \overline{\mu} \quad \text{in } \mathscr{M}(\Omega)$$

and thus

$$\begin{aligned}
(5.14) \qquad \limsup_{k\to\infty} \Big| \int_\Omega h_k\, dx \Big| &= \limsup_{k\to\infty} \Big| \int_\Omega f_k\, \psi_1(u_k - v)\, \varphi_1\, dx \Big| \\
&\leq \sup_{\mathbb{R}^m} |\psi_1| \langle \overline{\mu}, \varphi_1 \rangle\,.
\end{aligned}$$

Let x_0 be a point of approximate differentiability of u and a Lebesgue point of the measure $\overline{\mu}$ and the function $\overline{\sigma}$, i.e.,

$$(5.15) \qquad \limsup_{r\to 0} \frac{\overline{\mu}(B(x_0, r))}{r^n} < \infty,$$

$$(5.16) \qquad \limsup_{r\to 0} ⨍_{B(x_0,r)} |\overline{\sigma}(x) - \overline{\sigma}(x_0)|\, dx = 0.$$

In addition we may assume that x_0 is a Lebesgue point of the functions g_{ijlm}^M defined in (5.20) below. In order to localize the equation in x we define the rescaled cut-off functions

$$\varphi_r(x) = r^{-n} \varphi_1\Big(\frac{x - x_0}{r}\Big),$$

where $\varphi_1 \in C_0^\infty(B(0,1); \mathbb{R})$ is non-negative with $\int_{\mathbb{R}^n} \varphi_1 = 1$, and

$$\psi_r(x) = \frac{\alpha_r(x)}{|x|} x \qquad \text{with } \alpha_r(x) = r\, \alpha_1\Big(\frac{|x|}{r}\Big)\,.$$

Then inequality (5.12) holds for ψ_r, $r > 0$, since (5.8) is invariant under this scaling. Finally let

$$z = \frac{1}{r}(x - x_0)$$

denote the scaled coordinates around x_0 and let

$$\begin{aligned}
\tilde{u}_r(z) &= \frac{1}{r}(u(x) - v(x)), \\
\tilde{\sigma}_r(z) &= \overline{\sigma}(x).
\end{aligned}$$

Then (5.11), (5.13) and (5.14) yield

$$
\begin{aligned}
(5.17)\quad \mathrm{LHS}\,(r) &:= \int_\Omega \varphi_r(x) \int_{\mathbb{M}^{m\times n}} \sigma(x,u(x),\lambda) : (D\psi_r(u-v)(x)\lambda)\,d\nu_x(\lambda)\,dx \\
&\le r \sup|\psi_1| \,\sup\varphi_1\; r^{-n}\bar{\mu}(B(x_0,r)) \\
&\quad - \int_{B(0,1)} \bar{\sigma}(x_0+rz) : \psi_1(\tilde{u}_r(z)) \otimes D\varphi_1(z)\,dz \\
&\quad + \int_{B(0,1)} \varphi_1(z)\,\bar{\sigma}(x_0+rz) : D\psi_1(\tilde{u}_r(z)) Dv(rz+x_0)\,dz =: \mathrm{RHS}\,(r)\,.
\end{aligned}
$$

Choosing the function v as the first order Taylor approximation of u in x_0, i.e.,

$$v(x) = u(x_0) + \mathrm{ap}\, Du(x_0)(x-x_0)$$

we obtain by the approximate differentiability of u and (5.16) that for $r \to 0$

$$
\begin{aligned}
\tilde{u}_r &\to 0 && \text{in measure in } B(0,1),\\
\tilde{\sigma}_r &\to \bar{\sigma}(x_0) && \text{in } L^1(B(0,1)),
\end{aligned}
$$

and hence (at least for a subsequence)

$$
\begin{aligned}
&\psi_1 \circ \tilde{u}_r \to 0 && \text{boundedly almost everywhere,}\\
(5.18)\qquad &D\psi_1 \circ \tilde{u}_r \to \mathrm{Id} && \text{boundedly almost everywhere.}
\end{aligned}
$$

Thus we conclude

$$(5.19)\qquad \mathrm{RHS}\,(r) \to \bar{\sigma}(x_0) : \mathrm{ap}\, Du(x_0) \quad \text{as } r \to 0.$$

(and in fact the whole sequence $r \to 0$ converges as the approximate differential is independent of the sequence).

The passage to the limit $r \to 0$ on the left hand side of (5.17) is slightly more difficult since the functions g_{ijlm} defined by

$$g_{ijlm}(x) = \int_{\mathbb{M}^{m\times n}} \sigma_{ij}(x,u(x),\lambda)\lambda_{lm}\,d\nu_x(\lambda)$$

are in general not in $L^1(\Omega)$. The remedy here is to define the truncated functions

$$(5.20)\quad g^M_{ijlm}(x) = \int_{\mathbb{M}^{m\times n}} \eta\Big(\frac{|\lambda|}{M}\Big)\sigma_{ij}(x,u(x),\lambda)\lambda_{lm}\,d\nu_x(\lambda) \quad \text{for } M = 1,2,\ldots$$

where $\eta \in C_0^\infty(B(0,1);[0,1])$ denotes a fixed function satisfying $\eta \equiv 1$ on $B(0,\frac{1}{2})$. Note that for every fixed M the sequence $\sigma_{ij}(x,u_k(x),Du_k(x))(Du_k(x))_{lm} \times\eta(\frac{|Du_k(x)|}{M})$ is equiintegrable (since σ_k is equiintegrable) and therefore its weak L^1-limit is given by g^M_{ijlm}. By (5.2) we have

$$\sigma(x,u(x),\lambda) : D\psi(u(x)-v(x))\lambda \ge 0$$

and thus the left hand side in (5.17) is estimated by

$$\begin{aligned} \text{LHS }(r) &= \int_\Omega \varphi_r(x) g_{ijlj}(x)(D\psi_r)_{il}(u-v)(x)dx \\ &\geq \int_\Omega \varphi_r(x) g^M_{ijlj}(x)(D\psi_r)_{il}(u-v)(x)dx\,, \end{aligned} \tag{5.21}$$

where we take the sum over all repeated indices. Let $\tilde g_{ijlm,r}(z) = g_{ijlm}(x)$ and $\tilde g^M_{ijlm,r}(z) = g^M_{ijlm}(x)$ denote the rescaled functions as above. Since x_0 was chosen to be a Lebesgue point of g^M_{ijlm} we have

$$\tilde g^M_{ijlm,r}(z) \to g^M_{ijlm}(x_0) \quad \text{in } L^1(B(0,1)) \text{ for } r \to 0. \tag{5.22}$$

Using (5.21), (5.22), (5.18) and (5.19) we therefore obtain

$$\begin{aligned} \int_{\mathbb{M}^{m\times n}} \eta\Big(\frac{|\lambda|}{M}\Big)\sigma(x_0,u(x_0),\lambda) : \lambda\, d\nu_{x_0}(\lambda) &= g^M_{ijij}(x_0) \\ &\leq \bar\sigma(x_0) : \text{ap } Du(x_0) \end{aligned}$$

for all $M \in \mathbb{N}$. Choosing $b = 0$ in (5.2) we infer $\sigma(x_0,u(x_0),\lambda) : \lambda \geq 0$ and Lemma 11 follows by the monotone convergence theorem. □

6 Compactness and existence of solutions

In this section we use the div-curl inequality in Lemma 11 to show that the approximate solutions u_k constructed in Section 4 converge to a solution u of the equation (1.1). The key point here is to identify the weak limit $\bar\sigma = \sigma(\cdot,u,\text{ap } Du)$ of the sequence $\sigma_k = \sigma(\cdot,u_k,Du_k)$ and to prove the identity

$$\text{ap } Du(x) = \langle \nu_x, \text{Id}\,\rangle \qquad \text{for almost every } x \in \Omega, \tag{6.1}$$

where ν is the gradient Young measure generated by the sequence $\{Du_k\}_{k\in\mathbb{N}}$.

We first need the additional assumption (6.1). This will be later removed by Lemma 9 and the a priori estimates of Section 4.

Lemma 12 *Suppose that the sequence $\{u_k\}_{k\in\mathbb{N}}$ satisfies the hypotheses (5.1)–(5.7) and that the Young measure ν generated by the sequence $\{Du_k\}_{k\in\mathbb{N}}$ satisfies the identity (6.1). Assume that one of the following structure conditions holds:*

- *(i) σ is monotone and the mapping $F \mapsto \sigma(x,u,F)$ is continuously differentiable for all $(x,u) \in \Omega \times \mathbb{R}^m$.*
- *(ii) $\sigma(x,u,p) = \frac{\partial W}{\partial p}(x,u,p)$ and $p \mapsto W(x,u,p)$ is a convex C^1-function for all $(x,u) \in \Omega \times \mathbb{R}^m$.*
- *(iii) $p \mapsto \sigma(x,u,p)$ is strictly monotone for all $(x,u) \in \Omega \times \mathbb{R}^m$.*

Then $\bar\sigma(x) = \sigma(x,u(x),\text{ap } Du(x))$. If (ii) or (iii) holds then

$$\sigma(x,u_k(x),Du_k(x)) \to \sigma(x,u(x),\text{ap } Du(x)) \quad \text{strongly in } L^1(\Omega).$$

In case (iii) it follows in addition that $Du_k \to \text{ap } Du$ in measure.

Proof. We suppress throughout the proof the dependence on x and u, i.e., we write $\sigma(\lambda) = \sigma(x, u(x), \lambda)$ and $\nu = \nu_x$. Fix $x \in \Omega$ such that (6.1) and the conclusion of Lemma 11 hold and let $\bar\lambda = \langle \nu, \mathrm{Id} \rangle = \mathrm{ap}\, Du(x)$. We may assume by an affine transformation that $\bar\lambda = 0$ and $\sigma(\bar\lambda) = 0$. Then by Lemma 11

$$\int_{\mathbb{M}^{m\times n}} \sigma(\lambda) : \lambda \, d\nu(\lambda) \leq 0 \,.$$

By the monotonicity of σ we have

$$\sigma(\lambda) : \lambda \geq 0$$

whence

$$\sigma(\lambda) : \lambda = 0 \quad \text{on spt}\, \nu \tag{6.2}$$

and thus

$$\text{spt}\, \nu \subset \{\lambda \mid \sigma(\lambda) : \lambda = 0\}. \tag{6.3}$$

Case 1: Suppose that (i) holds. We claim that in this case the following identity holds on spt ν:

$$\sigma(\lambda) : \mu = -(D\sigma(0)\mu) : \lambda.$$

Indeed, by the monotonicity of σ we have for all $t \in \mathbb{R}$

$$(\sigma(\lambda) - \sigma(t\mu)) : (\lambda - t\mu) \geq 0,$$

whence

$$\begin{aligned} \sigma(\lambda) : \lambda - \sigma(\lambda) : (t\mu) &\geq \sigma(t\mu) : \lambda - \sigma(t\mu) : (t\mu) \\ &= (D\sigma(0)\mu) : (t\lambda) + o(t). \end{aligned}$$

The claim follows from this inequality using (6.3) since the sign of t is arbitrary. Thus

$$\begin{aligned} \bar\sigma &= \int_{\text{spt}\,\nu} \sigma(\lambda) d\nu(\lambda) = -(D\sigma(0))^t \int_{\text{spt}\,\nu} \lambda d\nu(\lambda) \\ &= -(D\sigma(0))^t \bar\lambda = 0 = \sigma(\bar\lambda). \end{aligned}$$

Case 2: Suppose that (ii) holds. We may assume in addition that $W(\bar\lambda) = 0$. We first show that the support of ν is contained in the set where W agrees with the supporting hyper-plane $W(\bar\lambda) + \sigma(\bar\lambda)(\lambda - \bar\lambda) \equiv 0$ in $\bar\lambda$:

$$\text{spt}\, \nu \subset K = \{\lambda \in \mathbb{M}^{m\times n} \,:\, W(\lambda) = 0\}.$$

If $\lambda \in \text{spt}\, \nu$ then by (6.3) $\sigma(\lambda) : \lambda = 0$ and it follows from the monotonicity of σ that $\sigma(t\lambda) : \lambda = 0$ for all $t \in [0, 1]$. Hence $W(\lambda) = \int_0^1 \sigma(t\lambda)\lambda dt = 0$ as claimed.

By the convexity of W we have $W(p) \geq 0$ for all $p \in \mathbb{M}^{m\times n}$ and thus $L \equiv 0$ is a supporting hyper-plane for all $\lambda \in K$. Since the mapping $p \mapsto W(p)$ is by assumption continuously differentiable we obtain

$$\sigma(\lambda) = 0 = \sigma(\overline{\lambda}) \quad \text{for all } \lambda \in K \supset \operatorname{spt} \nu \tag{6.4}$$

and thus

$$\overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma(\lambda)\, d\nu(\lambda) = \sigma(\overline{\lambda})\,. \tag{6.5}$$

Now consider the Carathéodory function

$$g(x,u,p) = |\sigma(x,u,p) - \overline{\sigma}(x)|\,.$$

The sequence $g_k(x) = g(x, u_k(x), Du_k(x))$ is by (5.4) equiintegrable and thus

$$g_k \rightharpoonup \overline{g} \quad \text{weakly in } L^1(\Omega)$$

and the weak limit $\overline{g}$ is given by

$$\begin{aligned} \overline{g}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,\eta,\lambda) - \overline{\sigma}(x)|\, d\delta_{u(x)}(\eta)\otimes d\nu_x(\lambda) \\ &= \int_{\operatorname{spt}\nu} |\sigma(x,u(x),\lambda) - \overline{\sigma}(x)|\, d\nu_x(\lambda) = 0 \end{aligned}$$

by (6.4) and (6.5). Since $g_k \geq 0$ it follows that

$$g_k \to 0 \quad \text{strongly in } L^1(\Omega)$$

and the proof of the second case is finished.

Case 3: Suppose that (iii) holds. In this case (6.2) implies by the strict monotonicity of σ that

$$\nu = \delta_{\overline{\lambda}} = \delta_{\mathrm{ap}\, Du(x)}.$$

Thus Du_k converges in measure to ap Du and the result follows by Vitali's convergence theorem using the equiintegrability (5.4) of σ_k. □

Lemma 13 *Suppose that the sequence $\{u_k\}_{k\in\mathbb{N}}$ satisfies the hypotheses (5.1)–(5.7) and the inequality*

$$\int_{|u_k|\leq\alpha} |Du_k|^p\, dx \leq C(\alpha).$$

Assume in addition that the Young measure ν generated by the sequence $\{Du_k\}_{k\in\mathbb{N}}$ is a $W^{1,p}$ gradient Young measure and that ν satisfies the identity (6.1). Assume finally that $\sigma(x,u,\cdot)$ is strictly p-quasimonotone for almost every $x \in \Omega$ and all $u \in \mathbb{R}^m$. Then $\overline{\sigma}(x) = \sigma(x,u(x),\mathrm{ap}\, Du(x))$, $Du_k \to \mathrm{ap}\, Du$ in measure and

$$\sigma(x,u_k(x),Du_k(x)) \to \sigma(x,u(x),\mathrm{ap}\, Du(x)) \quad \textit{strongly in } L^1(\Omega).$$

Proof. Since ν_x is for almost every $x \in \Omega$ a homogeneous gradient Young measure (see Lemma 9) we deduce from the definition of the strict p-quasimonotonicity of σ that

$$\begin{aligned}\int_{\mathbb{M}^{m\times n}} \sigma(x, u(x), \lambda)\lambda d\nu_x(\lambda) \geq \int_{\mathbb{M}^{m\times n}} \sigma(x, u(x), \lambda) d\nu_x(\lambda) \\ \times \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = \bar{\sigma}(x) \text{ap } Du(x).\end{aligned} \tag{6.6}$$

On the other hand Lemma 11 implies that in fact equality holds in (6.6) and thus by (6.1) $\nu_x = \delta_{\text{ap } Du(x)}$. The result follows now as in Case 3 in the proof of Lemma 12. □

Proof of Theorem 2 and Corollary 4. We give the proof first for the case that σ has the same growth and coercivity rate, i.e., $p = q$. The general result follows from this case using an approximation of σ.

Case 1: $p = q$. The solutions u_k of the approximate problems (4.1)

$$-\text{div } \sigma(x, u_k(x), Du_k(x)) = f_k(x) \qquad \text{in } \Omega \tag{6.7}$$

satisfy the a priori bounds (4.4)

$$\|Du_k\|^*_{L^{s,\infty}} + \|u_k\|^*_{L^{s^*,\infty}} \leq C(c_1, c_2, \|f_k\|_{L^1}, \text{meas } \Omega).$$

with s, $s^* > 0$ as well as the estimate (4.5)

$$\int_{|u|<\alpha} |Du_k|^p dx \leq C(\alpha).$$

In view of the embedding $L^{\beta,\infty} \hookrightarrow L^\alpha$ for $0 < \alpha < \beta$ the assumptions of Lemma 8 and Lemma 9 are satisfied. Thus there exists a measurable function $u : \Omega \to \mathbb{R}^m$ such that (for a subsequence) $u_k \to u$ in measure and $\langle \nu_x, \text{Id} \rangle = \text{ap } Du(x)$ for almost every $x \in \Omega$ where ν denotes the Young measure generated by the sequence $\{Du_k\}$. It follows from (H0), (H2) and (H3) that the hypothesis (5.1)–(5.7) in the div-curl inequality (Lemma 11) are fulfilled and by Lemma 12 and Lemma 13 we deduce the weak convergence of $\sigma(\cdot, u_k, Du_k)$ to $\sigma(\cdot, u, \text{ap } Du)$ in L^1. Thus we can pass to the limit in (6.7) and obtain

$$-\text{div } \sigma(\cdot, u, \text{ap } Du) = f \qquad \text{in } \mathscr{D}'(\Omega),$$

i.e., u is a solution of the equation in the sense of Definition 1. Note that if $p > 2 - \frac{1}{n}$ then the sequence $\{Du_k\}$ is equiintegrable and consequently ap $Du(x) = Du(x) = \langle \nu_x, \text{Id} \rangle$ for almost every $x \in \Omega$, i.e., u is a solution of the equation and ap Du agrees with the usual weak derivative of u. It remains to prove the a priori estimate (1.3) for u. Choose a cut-off function $\eta \in C_0^1(B(0, 2\alpha))$ such that $\eta \equiv \text{Id}$ on $B(0, \alpha)$ and $|D\eta| \leq C$ where C is independent of α. Since $\eta(u_k) \rightharpoonup \eta(u)$ in $W^{1,p}$ and ap $D(\eta \circ u) = \text{ap } Du$ on $\{|u| < \alpha\}$ (see the proof of Lemma 8) we deduce

$$\int_{\Omega} |D(\eta \circ u)|^p dx \leq \liminf_{k\to\infty} \int_{|u_k|\leq 2\alpha} |D\eta|^p |Du_k|^p dx \leq C(\alpha)$$

and thus

$$\int_{|u|<\alpha} |\text{ap } Du|^p dx \leq C(\alpha).$$

The estimate for u in the weak Lebesgue spaces follows now as in Section 4.

Case 2: The general case $p-1 \leq q-1 < \frac{n}{n-1}(p-1)$. The idea is to consider the regularized problems

$$-\text{div } \sigma_\varepsilon(x, u_\varepsilon(x), Du_\varepsilon(x)) = \mu \quad \text{in } \Omega, \tag{6.8}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{6.9}$$

with

$$\sigma_\varepsilon(x, u, F) := \sigma(x, u, F) + \varepsilon |F|^{s-2} F$$

for some $s > n+1$ and $\varepsilon < \frac{1}{2}$. Then σ_ε satisfies (H0)–(H3) with coercivity and growth rate both equal to s, i.e.,

$$\begin{aligned} \sigma_\varepsilon(x, u, F) : F &\geq \varepsilon |F|^s, \\ |\sigma_\varepsilon(x, u, F)| &\leq |F|^{s-1} + k(c_3, s, q). \end{aligned}$$

Using the results in Case 1 we find a solution $u_\varepsilon \in W_0^{1,s}(\Omega)$ of (6.8), (6.9). Testing (6.8) with u_ε yields

$$\varepsilon \int_{\Omega} |Du_\varepsilon|^s dx \leq \langle u_\varepsilon, \mu \rangle \leq \|u_\varepsilon\|_{L^\infty(\Omega)} \|\mu\|_{\mathscr{M}}.$$

Using Sobolev's embedding theorem

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C\, \|Du_\varepsilon\|_{L^s(\Omega)}$$

we conclude

$$\begin{aligned} \|Du_\varepsilon\|_{L^s(\Omega)} &\leq \left(\frac{C\, \|\mu\|_{\mathscr{M}}}{\varepsilon}\right)^{\frac{1}{s-1}}, \\ \|u_\varepsilon\|_{L^\infty(\Omega)} &\leq C\left(\frac{C\, \|\mu\|_{\mathscr{M}}}{\varepsilon}\right)^{\frac{1}{s-1}} \end{aligned}$$

and

$$\|\varepsilon |Du_\varepsilon|^{s-1}\|_{L^{\frac{s}{s-1}}(\Omega)} \leq C \|\mu\|_{\mathscr{M}}. \tag{6.10}$$

By testing equation (6.8) with $T_\alpha(u_\varepsilon)$ we obtain, as in Lemma 10, that

$$\|Du_\varepsilon\|^*_{L^{\frac{n}{n-1}(p-1),\infty}(\Omega)} \leq C$$

for a constant C which does not depend on ε. Thus, in combination with (6.10), we have that for all $\rho < \frac{s}{s-1}$

$$\lim_{\varepsilon\to 0} \|\varepsilon\,|Du_\varepsilon|^{s-1}\|_{L^{\rho}(\Omega)} = 0$$

and hence in particular

$$\|\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) - \sigma_0(x,u_\varepsilon,Du_\varepsilon)\|_{L^1(\Omega)} \to 0 \quad \text{as } \varepsilon\to 0. \tag{6.11}$$

Thus the weak L^1-limit $\bar\sigma_0$ of the sequence $\sigma_0(\cdot,u_\varepsilon,Du_\varepsilon)$ satisfies the equation

$$-\text{div}\,\bar\sigma_0 = \mu \text{ in } \mathscr{D}'(\Omega).$$

If we test (6.8) with $\psi\circ(u_\varepsilon - v)\,\varphi$ (ψ and φ as in Section 5) we obtain

$$\begin{aligned}\sup_\Omega |\psi|\,\langle|\mu|,\varphi\rangle \geq &\\ \int_\Omega \big(&\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Du_\varepsilon\varphi\\ &-\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Dv\varphi\\ &+\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : \psi(u_\varepsilon - v)\otimes D\varphi\big)\,dx\,.\end{aligned}$$

By definition

$$\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Du_\varepsilon\varphi \geq \sigma_0(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Du_\varepsilon\varphi$$

and thus (6.11) implies

$$\begin{aligned}\sup_\Omega |\psi|\,\langle|\mu|,\varphi\rangle \geq &\\ \limsup_{\varepsilon\to 0}\int_\Omega \big(&\sigma_0(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Du_\varepsilon\varphi\\ &-\sigma_0(x,u_\varepsilon,Du_\varepsilon) : (D\psi)(u_\varepsilon - v)Dv\varphi\\ &+\sigma_0(x,u_\varepsilon,Du_\varepsilon) : \psi(u_\varepsilon - v)\otimes D\varphi\big)\,dx\,.\end{aligned}$$

Since $q-1 < \frac{n}{n-1}(p-1)$, the sequence $\sigma_0(x,u_\varepsilon,Du_\varepsilon)$ is equiintegrable in $L^1(\Omega)$ and the arguments in Section 5 apply. $\square$

7 The critical case $p = n$

In this section we prove that solutions of the elliptic system (1.1) are bounded in $BMO_{\text{loc}}(\Omega)$ for $p = q = n$. Our proof is strongly inspired by Simon's beautiful proof of $C^{0,\alpha}$ estimates for the Poisson equation by scaling and compactness (see [Si1]). Here we say that $u \in BMO_{\text{loc}}(\Omega)$ if $u \in L^1_{\text{loc}}(\Omega)$ and for all open $U \subset\subset \Omega$ there exists a constant $C(U)$ such that

$$[\,u\,]^n_{BMO(U,\Omega)} = \sup_{y\in\bar U}\ \sup_{Q(y,R)\subset\Omega} \frac{1}{R^n}\int_{Q(y,R)} |u(x) - u_{y,R}|^n\,dx \leq C(U),$$

where $u_{y,R}$ denotes the mean value of u on the cube $Q(y,R)$. In Lemma 14 we first show a localized version of the a priori bound (4.5) for solutions $u \in W^{1,n}_0(\Omega;\mathbb{R}^m)$ of the approximating system

$$-\operatorname{div}\sigma(x,u(x),Du(x))=f \tag{7.1}$$

with $f\in L^1(\Omega;\mathbb{R}^m)$. Since such a result does not seem to hold for $q>n$ we restrict ourselves to the case $q=n=p$ in this section.

Lemma 14 *Let $u\in W^{1,n}(\Omega;\mathbb{R}^m)$ be a solution of system (7.1) with $f\in L^1(\Omega;\mathbb{R}^m)$. Then there exist constants C_0, C_1 such that the inequality*

$$\begin{aligned}&\int_{\substack{|u-\beta|<\alpha\\ \cap Q(y,R/2)}}|Du|^n dx\\ &\le\frac{C_0}{R^n}\int_{Q(y,R)\setminus Q(y,R/2)}|u-\beta|^n dx+C_1(\alpha\|f\|_{L^1(\Omega)}+R^n)\end{aligned} \tag{7.2}$$

holds for all cubes $Q(y,R)\subset\Omega$ and all $\beta\in\mathbb{R}^m$.

Proof. Let $\eta\in C_0^\infty(\Omega)$ be a cut-off function such that $\eta\equiv 1$ on $Q(y,R/2)$, $0\le\eta\le 1$ and $|D\eta|\le C/R$. Choose a smooth function $a_\alpha:\mathbb{R}\to\mathbb{R}$ with the following properties: $a_\alpha\equiv\mathrm{Id}$ on $[0,\alpha]$, $0\le a_\alpha\le n\alpha$, $a'_\alpha\le 1$ and

$$0<c\left(\frac{a_\alpha(s)}{s}\right)^{n/(n-1)}\le a'_\alpha(s)\le\frac{a_\alpha(s)}{s}\qquad\text{on }(0,\infty). \tag{7.3}$$

A possible choice is

$$a_\alpha(s)=\begin{cases} s & \text{for } s\le\alpha,\\ \alpha+\displaystyle\int_\alpha^s\left(\frac{\alpha}{t}\right)^{n/(n-1)}dt & \text{for } s>\alpha.\end{cases}$$

Define the cut-off function φ_α in the target by

$$\varphi_\alpha(z)=\frac{a_\alpha(|z|)}{|z|}\,z\,.$$

Then

$$D(\varphi_\alpha\circ v)=\frac{a_\alpha(|v|)}{|v|}\left(\mathrm{Id}-\frac{v}{|v|}\otimes\frac{v}{|v|}\right)Dv+a'_\alpha(|v|)\left(\frac{v}{|v|}\otimes\frac{v}{|v|}\right)Dv,$$

and by (7.3), (H2) and (H3)

$$\sigma(Dv):D(\varphi_\alpha\circ v)\ge\sigma(Dv):Dv\,a'_\alpha(|v|)\ge a'_\alpha(c_1|Dv|^n-c_2).$$

Testing the equation (7.1) with $\eta^n\varphi_\alpha\circ(u-\beta)$ we obtain

$$\begin{aligned}&\int_\Omega\eta^n\sigma(Du):D[\varphi_\alpha\circ(u-\beta)]dx=\\ &\quad=-\int_\Omega n\eta^{n-1}\sigma(Du):\varphi_\alpha\circ(u-\beta)\otimes D\eta dx+\int_\Omega\eta^n f\varphi_\alpha\circ(u-\beta)dx\,.\end{aligned}$$

It follows by (H2) with $p=q=n$ and by using Hölder's inequality and (7.3) on the right hand side that

$$\int_\Omega \eta^n |Du|^n a'_\alpha(|u-\beta|)\,dx$$
$$\leq \frac{C}{R}\left(\int_{Q(y,R)} \eta^n (c_3|Du|+c_2)^n a'_\alpha(|u-\beta|)\,dx\right)^{(n-1)/n} \times$$
$$\times \left(\int_{Q(y,R)\setminus Q(y,R/2)} |u-\beta|^n dx\right)^{1/n} + C\alpha\|f\|_{L^1(\Omega)} + CR^n .$$

Application of Young's inequality yields

$$\begin{aligned}\int_\Omega \eta^n a'_\alpha(|u-\beta|)|Du|^n dx \quad &\leq \quad \frac{C_0}{R^n}\int_{Q(y,R)\setminus Q(y,R/2)} |u-\beta|^n dx \\ &\quad + C_1(\alpha\|f\|_{L^1(\Omega)} + R^n),\end{aligned}$$

and inequality (7.2) follows from the definition of a_α. □

The following lemma shows that a function satisfying an inequality like (7.2) is a function of locally bounded mean oscillation.

Lemma 15 *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in W^{1,n}(\Omega;\mathbb{R}^m)$ and suppose that the estimate*

$$(7.4)\ \int_{\substack{|u-\beta|<\alpha \\ \cap Q(y,R/2)}} |Du|^n dx \leq \frac{C_0}{R^n}\int_{Q(y,R)\setminus Q(y,R/2)} |u-\beta|^n dx + C_1(\alpha + R^n)$$

holds for all cubes $Q(y,R) \subset \Omega$ and all $\beta \in \mathbb{R}^m$. Then $u \in BMO_{\mathrm{loc}}(\Omega)$ and

$$[\,u\,]_{BMO(U,\Omega)} \leq C_2\,(1 + \|u\|_{L^n(\Omega)}),$$

where C_2 depends only on C_0, C_1, and U.

Proof. It suffices to show that $\|u\|_{L^n(\Omega)} \leq 1$ implies $[u]_{BMO(U,\Omega)} \leq C_2$. Indeed, if $\|u\|_{L^n(\Omega)} > 1$, observe that $\tilde u := \frac{u}{1+\|u\|_{L^n(\Omega)}}$ satisfies (7.4) and that the estimate $[\tilde u]_{BMO(U,\Omega)} \leq C_2$ implies the assertion. Now suppose for a contradiction that there is a sequence u_k such that $\|u_k\|_{L^n(\Omega)} \leq 1$ and $[u_k]_{BMO(U,\Omega)} \to \infty$ for $k \to \infty$. Thus there exist $x_k \in \overline{U}$ and $r_k > 0$ such that $Q(x_k,r_k) \subset \Omega$ and

$$\frac{1}{r_k^n}\int_{Q(x_k,r_k)} |u_k - (u_k)_{x_k,r_k}|^n\,dx \geq \frac{1}{2}[\,u_k\,]_{BMO(U,\Omega)}.$$

We deduce that $r_k \to 0$ since $\|u_k\|_{L^n(\Omega)} \leq 1$. Define the rescaled functions $v_k : \Omega_k = \frac{1}{r_k}(-x_k + \Omega) \to \mathbb{R}^m$ by

$$v_k(z) = \frac{u_k(x_k + r_k z) - (u_k)_{x_k,r_k}}{[\,u_k\,]_{BMO(U,\Omega)}}\,,$$

and let $U_k = \frac{1}{r_k}(-x_k + U)$. Then $v_k \in BMO_{\mathrm{loc}}(\Omega_k)$, $[\,v_k\,]_{BMO(U_k,\Omega_k)} = 1$,

$$\int_{Q(0,1)} v_k\,dx = 0,$$

and

$$\int_{Q(0,1)} |v_k - (v_k)_{0,1}|^n \, dx \geq \frac{1}{2}\,.$$

Using (7.4) we obtain the following inequality for the rescaled functions v_k:

$$\int_{\substack{|v_k - \beta| < \alpha \\ \cap Q(y,R/2)}} |Dv_k|^n dx \leq \frac{C_0}{R^n} \int_{Q(y,R)\setminus Q(y,R/2)} |v_k - \beta|^n dx + \frac{C_1(\alpha + (r_k R)^n)}{[\, u_k \,]^{n-1}_{BMO(U,\Omega)}}$$

for all $Q(y, R) \subset \Omega_k$ and all $\beta \in \mathbb{R}^m$. We claim that the sequence $\{v_k\}$ is bounded in $W^{1,s}_{\mathrm{loc}}$ for all $s < n$. To see this, fix $R_0 > 0$ and choose k_0 big enough such that $Q(0, 2R_0) \subset \Omega_k$ for all $k \geq k_0$. Choosing $y = 0$ and $\beta = (v_k)_{0,2R_0}$ we obtain from the inequality above

$$\int_{\substack{|v_k - (v_k)_{0,2R_0}| < \alpha \\ \cap Q(0,R_0)}} |Dv_k|^n dx \leq C[\, v_k \,]^n_{BMO(U_k,\Omega_k)} + \frac{C_1(\alpha + (r_k R_0)^n)}{[\, u_k \,]^{n-1}_{BMO(U,\Omega)}}.$$

Since

$$|(v_k)_{0,2R_0}| = |(v_k)_{0,2R_0} - (v_k)_{0,1}| \leq C(|\ln R_0| + 1)[\, v_k \,]_{BMO(U_k,\Omega_k)} \leq C(|\ln R_0| + 1)$$

we deduce with $\gamma(t) = C(|\ln t| + 1)$

$$\int_{\substack{|v_k| < \alpha - \gamma(R_0) \\ \cap Q(0,R_0)}} |Dv_k|^n dx \leq C + \frac{C_1(\alpha + (r_k R_0)^n)}{[\, u_k \,]^{n-1}_{BMO(U,\Omega)}}.$$

This implies

$$\int_{\substack{|v_k| < \alpha/2 \\ \cap Q(0,R_0)}} |Dv_k|^n dx \leq C + C(\alpha + (r_k R_0)^n) \qquad \text{for } \alpha \geq 2\gamma(R_0)$$

and the idea is to use the methods in Section 4 to bound v_k and Dv_k in the weak Lebesgue spaces $L^{p^*,\infty}$ and $L^{p,\infty}$ for all $p < n$. Define the truncation function T_α as in Section 4 by $T_\alpha(y) = \min\{1, \frac{\alpha}{|y|}\}y$. From $|(v_k)_{0,R_0}| < C(R_0)$ and $[\, v_k \,]_{BMO(U_k,\Omega_k)} \leq 1$ we deduce

$$\begin{aligned}
\int_{Q(0,R_0)} |T_\alpha(v_k)|^n dx &\leq \int_{Q(0,R_0)} |v_k|^n dx \\
&\leq C \int_{Q(0,R_0)} |v_k - (v_k)_{0,R_0}|^n dx + C \int_{Q(0,R_0)} |(v_k)_{0,R_0}|^n dx \\
&\leq CR_0^n + C(R_0)R_0^n
\end{aligned}$$

and

$$
\begin{aligned}
\int_{Q(0,R_0)} |D(T_\alpha(v_k))|^n dx &\leq \int_{\substack{|v_k| < \alpha \\ \cap Q(0,R_0)}} |Dv_k|^n dx \\
&\quad + \sum_{l=0}^{\infty} \int_{\substack{2^l\alpha \leq |v_k| < 2^{l+1}\alpha \\ \cap Q(0,R_0)}} |D(T_\alpha(v_k))|^n dx \\
&\leq \int_{\substack{|v_k| < \alpha \\ \cap Q(0,R_0)}} |Dv_k|^n dx \\
&\quad + C \sum_{l=0}^{\infty} \frac{1}{2^{ln}} \int_{\substack{|v_k| < 2^{l+1}\alpha \\ \cap Q(0,R_0)}} |Dv_k|^n dx \\
&\leq C + C(\alpha + (r_k R_0)^n).
\end{aligned}
$$

Sobolev's embedding theorem yields for all $p < n$

$$
\begin{aligned}
\int_{\substack{|v_k| < \alpha \\ \cap Q(0,R_0)}} |v_k|^{p^*} dx &\leq \left(\int_{Q(0,R_0)} |T_\alpha(v_k)|^p + |D(T_\alpha(v_k))|^p \right)^{p^*/p} \\
&\leq C(R_0)(\alpha^{p^*/n} + 1).
\end{aligned}
$$

Using the same arguments as in Section 4 we obtain for all $s < n$

$$
\|v_k\|_{W^{1,s}(Q(0,R_0))} \leq C(R_0, s).
$$

In particular there exists a subsequence (not relabeled) such that

$$
v_k \to v \quad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^n) \text{ for all } q < \infty
$$

and

$$
\int_{Q(0,1)} v\, dx = 0. \tag{7.5}
$$

Choose a cut-off function $\varphi_\alpha \in C_0^\infty(B(0,2\alpha))$ such that $\varphi_\alpha \equiv \mathrm{Id}$ on $B(0,\alpha)$ and $|D\varphi_\alpha| \leq \overline{C}$ where the constant $\overline{C}$ is independent of α. Then $\varphi_\alpha \circ (v_k - \beta)$ is bounded in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$ and converges to $\varphi_\alpha \circ (v - \beta)$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for all $p < \infty$ while $D(\varphi_\alpha \circ (v_k - \beta))$ converges weakly in $L^n_{\mathrm{loc}}(\mathbb{R}^n)$ to $D(\varphi_\alpha \circ (v - \beta))$. By the lower semicontinuity of the L^n-norm we obtain

$$
\begin{aligned}
\int_{\substack{|v-\beta|<\alpha \\ \cap Q(y,R_0/2)}} |Dv|^n\,dx &\le \int_{Q(y,R_0/2)} |D(\varphi_\alpha \circ (v-\beta))|^n\,dx \\
&\le \liminf_{k\to\infty} \int_{Q(y,R_0/2)} |D(\varphi_\alpha \circ (v_k-\beta))|^n\,dx \\
&\le \bar{C} \liminf_{k\to\infty} \int_{\substack{|v-\beta|<2\alpha \\ \cap Q(y,R_0/2)}} |Dv_k|^n\,dx \\
&\le \bar{C} C_0 \liminf_{k\to\infty} \left\{ \frac{1}{R_0^n} \int_{Q(y,R_0)\setminus Q(y,R_0/2)} |v_k-\beta|^n\,dx + \frac{\bar{C} C_1(\alpha+(r_k R_0)^n)}{[\,u_k\,]^{n-1}_{BMO(U,\Omega)}} \right\} \\
&= \bar{C} C_0 \frac{1}{R_0^n} \int_{Q(y,R_0)\setminus Q(y,R_0/2)} |v-\beta|^n\,dx\,.
\end{aligned}
$$

Using the monotone convergence theorem we may pass to the limit $\alpha \to \infty$ and get

$$
\int_{Q(y,R/2)} |Dv|^n\,dx \le \bar{C} C_0 \frac{1}{R^n} \int_{Q(y,R)\setminus Q(y,R/2)} |v-\beta|^n\,dx\,. \tag{7.6}
$$

If we choose $\beta = (v)_{y,R}$ then the right hand side in this inequality is estimated by $[\,v\,]_{BMO} \le 1$ since the BMO-norm is lower semicontinuous. Thus $Dv \in L^n(\mathbb{R}^n)$. Application of Poincaré's inequality to the right hand side of (7.6) shows

$$
\int_{Q(y,R/2)} |Dv|^n\,dx \le C \int_{Q(y,R)\setminus Q(y,R/2)} |Dv|^n\,dx\,,
$$

where C is independent of R. It follows for $R \to \infty$ that $Dv \equiv 0$ and in view of (7.5) that $v \equiv 0$. On the other hand the strong convergence of v_k in L^n implies that

$$
\frac{1}{2} \le \int_{Q(0,1)} |v_k - (v_k)_{0,1}|^n\,dx \to \int_{Q(0,1)} |v - (v)_{0,1}|^n\,dx
$$

This is a contradiction and the lemma is proven. $\square$

Theorem 16 *Assume that the hypotheses in Theorem 2 are satisfied with $p = q = n$. Then the system (1.1), (1.2) has a solution $u \in BMO_{\mathrm{loc}}(\Omega;\mathbb{R}^m) \cap W_0^{1,s}(\Omega;\mathbb{R}^m)$ for all $s < n$ in the sense of Definition 1 and the a priori estimate*

$$
\|u\|_{BMO(U,\Omega)} + \|u\|_{W^{1,s}(\Omega)} \le C(s, U, \|\mu\|_{\mathscr{M}})
$$

holds for all $s < n$ and all open $U \subset\subset \Omega$.

Proof. Consider the solutions u_k of the approximating system (4.1), (4.2). Using the same methods as in Section 4 one obtains $\|u_k\|_{W^{1,s}(\Omega)} \le C(s)$ for all $s < n$ and thus u_k converges weakly to u in $W^{1,s}(\Omega)$ for all $s < n$ and strongly in $L^p(\Omega)$ for all $p < \infty$. By Lemmata 14 and 15

$$
[\,u_k\,]_{BMO(U,\Omega)} \le C(U, \|\mu\|_{\mathscr{M}})
$$

for all open $U \subset \Omega$ and due to the strong convergence of u_k in L^n we may pass to the limit in this inequality. $\square$

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Added in proof. We recently proved that in the critical case $p = q = n$ the solution constructed here satisfies $Du \in L^{n,\infty}(\Omega)$, provided $\mathbb{R}^n \setminus \Omega$ is of type A. We also establish certain uniqueness results in this class.