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CONSTANT SUMS OF DISTANCES TO VERTICES AND SIDES OF POLYGONS

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Abstract. This paper is a follow-up paper to [7], where squares instead of higher powers are discussed. The following investigation was stimulated by a post by David Vreken on 27 June 2026 in the 'Romantics of Geometry' group on Facebook about the constant sum of 4th powers of the distances from a point P on the incircle of an equilateral triangle to the vertices. Here we initially focus on a special constant sum of the fourth powers of the distances from a point P to the *vertices* (A) of a polygon and provide a proof. Furthermore, this result can be generalized with the help of complex numbers to other even powers (for odd powers this result does not hold). Then, we discuss the case of the sum of the fourth powers of distances to the *sides* (B) of a regular n-gon with $n \geq 5$ and give corresponding generalizations. Surprisingly, there is a structural difference between (A) and (B), in case of (A) odd powers do not play any role, whereas they do for (B) under specific restrictions. Furthermore, the mentioned specific restrictions may surprise at first glance, and they indeed are not necessary for the case of oriented (signed) instead of Euclidean distances. Finally, it was a big surprise for us that in Theorem 7 also for $m = n$ a perfect cancellation takes place for semi-regular polygons.

1. Introduction

Constant power sums of geometric quantities—such as distances, radii, or edge lengths—are a recurring topic in geometry. The most elementary example is the famous Pythagorean theorem: for any point on a circle, the sum of the squared distances to two diametrically opposed points is constant, equaling the square of the circle's diameter. More generally, if a point C is moving on a circle centered at the midpoint of a line segment AB (not necessarily with radius $\frac{AB}{2}$ as it is the case for a right triangle ΔABC) then $CA^2 + CB^2$ is constant as well (see, e.g., [6]).

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As a further generalization step, it is a classical result that the sum of squared distances from a point P to a set of n points in Euclidean space depends solely on the distance from P to the barycenter of those points (see, e.g., [1, p. 70–71] and [2]).

When higher symmetries are present, this invariance extends beyond quadratic expressions. For a regular n-gon centered at O, the sum of the $2m^{\text{th}}$ powers of the distances from a point P to the vertices remains constant as P moves along any circle centered at O, provided that $m < n$ is an integer (see [10, 9], where extensions to Platonic solids in three dimensions are also treated).

Analogous conservation laws appear in parametric families of geometric objects and higher-dimensional projections. In a one-parameter family of Steiner chains of length k, the power sum of inverse radii $\sum_{j=1}^{k} r_j^{-m}$ is invariant for all $m \in \{1, 2, \ldots, k-1\}$ (see [11]). Similarly, when projecting an edge-transitive N-dimensional polytope onto an M-dimensional subspace of $\mathbb{R}^N$, the ratio between the sums of squared lengths of the original and projected edges is constant and equals N/M [8]. Another classical result is Viviani's theorem and its generalizations to broader classes of polygons. These state that the sum of the (signed) distances from a point to the sides is constant for equiangular or equilateral polygons, as well as $2n$-gons with opposite sides parallel.

The aforementioned result regarding the power sums of distances to the vertices of a regular polygon is strictly limited to even powers. Viviani's theorem, on the other hand, shifts the focus to the distances to the sides of a polygon. In the present article, we bridge these concepts by investigating whether higher power sums of distances to the sides of a regular n-gon remain constant. Surprisingly, for points moving along a concentric circle, this invariance holds true—subject to certain restrictions—even for *odd* powers $m < n$. Furthermore, when considering semi-regular $2n$-gons, a remarkable cancellation phenomenon occurs, extending this geometric invariance to the threshold case $m = n$.

We invite readers to verify the results presented in this article and to experiment with them using dynamic geometry software on the website [5].

Taking an equilateral triangle ΔABC and a moving point P on its circumcircle, one can experimentally observe that the sum of the fourth powers of the distances from P to the vertices A, B, C, i.e., $PA^4 + PB^4 + PC^4$, is constant (see Figure 1, top).

Remarks:

- Assuming that the observation of Figure 1, top, is correct, the value of this constant sum can be easily determined in the case when P coincides with a vertex, namely $2a^4 = 18R^4$, where a is the side length of the triangle and R its circumradius.
- In [7] the analogous question—including generalizations—with exponent 2 instead of 4 is discussed.

Let us formulate this as a theorem more generally, not restricted to the circumcircle.

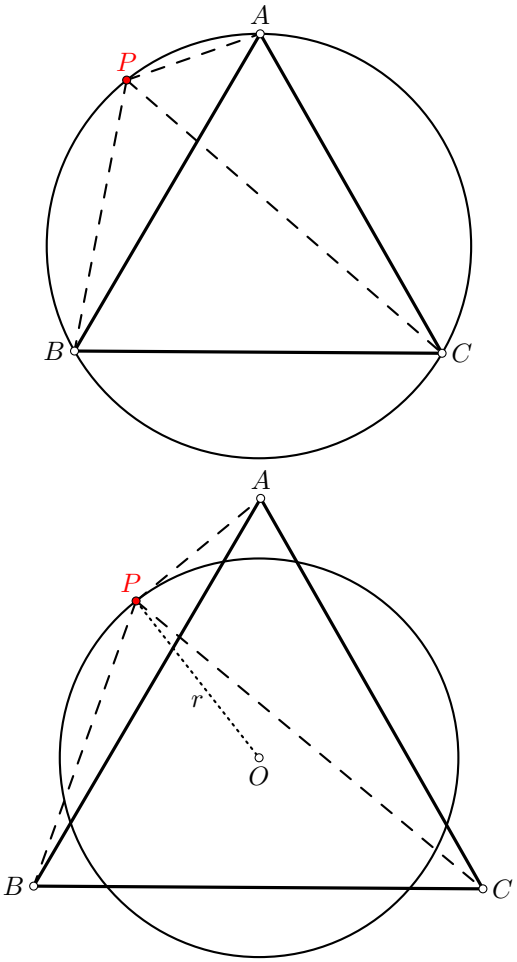


FIGURE 1. Top: $P \in$ circumcircle $\Rightarrow PA^4 + PB^4 + PC^4$ is constant. Bottom: $P \in$ circle with center O and radius r $\Rightarrow PA^4 + PB^4 + PC^4$ is constant

Theorem 1. *Let ΔABC be an equilateral triangle with circumradius R, circumcenter O, and P an arbitrary point on a circle centered at O with radius r. Then $PA^4 + PB^4 + PC^4$ is constant, namely $3(R^4 + 4R^2r^2 + r^4)$ (Figure 1, bottom).*

In Figure 1 (top) we had the special case $r = R$, where this formula yields $18R^4$. In the special case of the *incircle* we have the inradius ϱ as r and $R = 2\varrho$, yielding $PA^4 + PB^4 + PC^4 = 3(16\varrho^4 + 16\varrho^4 + \varrho^4) = 99\varrho^4$.

Theorem 1 is a special case of Theorem 3, thus, for brevity we do not give a proof here. For another proof (using sums of cosine powers) see [10, 9].

Let us take a step back to *squared* distances instead of distances to the fourth, but instead of an equilateral triangle we take more generally a regular n-gon $A_1 \dots A_n$ centered at the origin (circumradius R).

Theorem 2. *Let $A_1 \dots A_n$ be a regular n-gon with circumradius R, circumcenter O, and P an arbitrary point on a circle centered at O with radius r. Then $(PA_1)^2 + \dots + (PA_n)^2$ is constant, namely $n(R^2 + r^2)$.*

Proof. Using complex numbers[1], for $k = 1, \ldots, n$ we have with $\bar{z}$ denoting the complex conjugate of z:

$$\begin{aligned} (PA_k)^2 &= ||A_k - P||^2 = (A_k - P)\overline{(A_k - P)} = (A_k - P)\left(\overline{A_k} - \overline{P}\right) \\ &= \underbrace{A_k\overline{A_k}}_{R^2} - A_k\overline{P} - \overline{A_k}P + \underbrace{P\overline{P}}_{r^2} = R^2 + r^2 - A_k\overline{P} - \overline{A_k}P \end{aligned} \tag{1}$$

Now we must form the following sum which easily proves to be constant (note that the center of the regular n-gon is 0, thus $\sum_{k=1}^{n} A_k = 0 = \sum_{k=1}^{n} \overline{A_k}$), finishing the proof:

$$\sum_{k=1}^{n}(PA_k)^2 = nR^2 + nr^2 - \overline{P}\underbrace{\sum_{k=1}^{n} A_k}_{0} - P\underbrace{\sum_{k=1}^{n}\overline{A_k}}_{0} = \underbrace{n(R^2 + r^2)}_{\text{constant}} \qquad \square$$

In Theorem 1, the squared distances can be replaced by the fourth powers.

Theorem 3. *Let $A_1 \ldots A_n$ be a regular n-gon ($n \geq 3$) with circumradius R, circumcenter O, and P an arbitrary point on a circle centered at O with radius r. Then $(PA_1)^4 + \ldots + (PA_n)^4$ is constant, namely $n(R^4 + 4R^2r^2 + r^4)$.*

Proof. We will use complex numbers and relation (1). Without loss of generality, we may assume that A_n lies on the positive real axis (like in the case of the n^{th} roots of unity), and for brevity we set $c := R^2 + r^2$:

$$\begin{aligned} (PA_k)^4 &= \left(c - A_k\overline{P} - \overline{A_k}P\right)^2 = \\ &= c^2 + A_k^2\overline{P}^2 + \overline{A_k}^2P^2 - 2cA_k\overline{P} - 2c\overline{A_k}P + 2\underbrace{A_k\overline{A_k}}_{R^2}\underbrace{P\overline{P}}_{r^2} \end{aligned}$$

Note for the following that if the complex numbers A_k ($k = 1, \ldots, n$) form a regular[2] polygon centered at 0 then A_k^2 ($k = 1, \ldots, n$) form a regular polygon as well (for even n each vertex is "hit" twice—not affecting the following argumentation; roughly speaking, doubling all angles).

We must evaluate the following sum which easily proves to be constant as well (note that the center of the regular n-gon is 0, thus $\sum_{k=1}^{n} A_k = 0 = \sum_{k=1}^{n}\overline{A_k}$ like above, and additionally $\sum_{k=1}^{n} A_k^2 = 0 = \sum_{k=1}^{n}\overline{A_k}^2$), finishing the proof:

$$\sum_{k=1}^{n}(PA_k)^4 = n\underbrace{c^2}_{\substack{=(R^2+r^2)^2= \\ R^4+2R^2r^2+r^4}} + 2nR^2r^2 = \underbrace{n(R^4 + 4R^2r^2 + r^4)}_{\text{constant}} \qquad \square$$

The following generalization is immediate.

Theorem 4. *Let $A_1A_2 \ldots A_{2n-1}A_{2n}$ be a semi-regular $2n$-gon with $n \geq 3$ and P an arbitrary point on a circle with radius r centered at the semi-regular $2n$-gon's center. Then $(PA_1)^4 + \ldots + (PA_{2n})^4$ is constant.*

[1] Another proof is given in [7].

[2] With one vertex at the real axis.

Proof. There are two kinds of semi-regular polygons, namely on the one hand **semi-regular angle-gons** (equi-angular cyclic $2n$-gons, Figure 2, top), and on the other hand **semi-regular side-gons** (equilateral circumscribed $2n$-gons, Figure 2, bottom), see [4] and [3]. Note that in semi-regular polygons $A_1A_2 \ldots A_{2n-1}A_{2n}$ the n-gons $A_1A_3 \ldots A_{2n-1}$ and $A_2A_4 \ldots A_{2n}$ are regular sharing their coinciding centers (O, I), and that in case of semi-regular angle-$2n$-gons the mentioned regular n-gons have different incircles, and in case of semi-regular side-$2n$-gons different circumcircles. Hence, by Theorem 3 we have in both cases two constants, namely $(PA_1)^4 + \ldots + (PA_{2n-1})^4$ and $(PA_2)^4 + \ldots + (PA_{2n})^4$, and the sum of two constants is constant. □

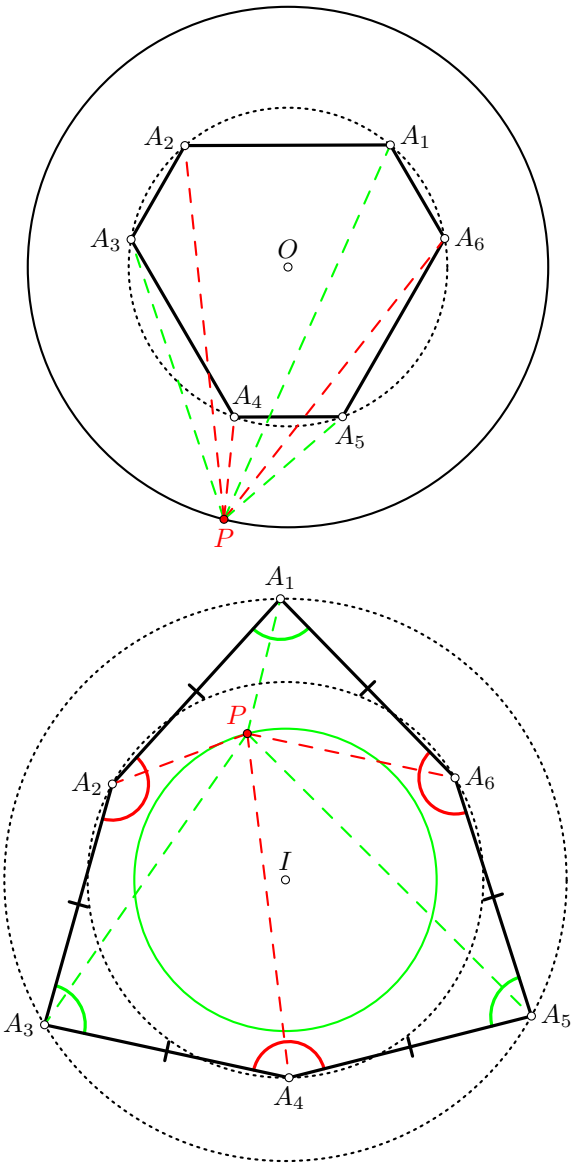


FIGURE 2. Top: Semi-regular angle-hexagon.
Bottom: Semi-regular side-hexagon.

Since the powers 2 and 4 of the mentioned distances gave interesting results, it is natural to explore the corresponding power sums with other even powers. Indeed, the following theorem holds.

Theorem 5. *Let $A_1 \dots A_n$ be a regular n-gon and P a point on a circle at the n-gon's center. Then for $n > m$ the sum $(PA_1)^{2m} + \dots + (PA_n)^{2m}$ is constant.*

Although this theorem is proved in [10, 9] by trigonometric means (sums of powers of cosine values) we will give here a proof based on complex numbers. Fortunately, the above idea of using complex numbers is strong enough to cover this generalization, as can also be read at [12], written by a user called "Tommy".

Lemma. Let z_k $(1 \le k \le n)$ be one of the n^{th} roots of unity. Then $\sum_{k=1}^{n} z_k^i = 0$ for $1 \le i < n$[3].

Proof.

Case 1: $\gcd(i, n) = 1$; then the values $z_1^i, z_2^i, \dots, z_n^i$ are different and represent all the n^{th} roots of unity (possibly in another order than $z_1, z_2, \dots, z_n$). Thus, the sum in the claim vanishes.

Case 2: $\gcd(i, n) = d > 1$; then the values $z_1^i, z_2^i, \dots, z_n^i$ are not all different, they represent the $\left(\frac{n}{d}\right)^{\text{th}}$ roots of unity, but each of them d times. As an example, take e.g. $n = 12$ and $i = 3 = d$. Then in $z_1^3, z_2^3, \dots, z_{12}^3$ (every angle is tripled) each of the four 4^{th} roots of unity is represented ("hit") three times. Thus, the sum in the claim vanishes as well. □

Proof of Theorem 5. By scaling and rotation, we can assume without loss of generality that the polygon's vertices $A_1, \dots, A_n$ are the n^{th} roots of unity $z_1, z_2, \dots, z_n$. From (1) with $c := R^2 + r^2$ we get

$$(PA_k)^{2m} \overset{A_k = z_k}{=} (Pz_k)^{2m} \overset{(1)}{=} (c - z_k\overline{P} - \overline{z_k}P)^m.$$

When expanding this fully and doing the summation over all n roots, note that c^m is constant. Furthermore, $z_k\overline{z_k} = 1$ and $P\overline{P} = r^2$, thus all not constant terms in the sum may be expressed as a constant$\times z_k^i \overline{P}^i$ (or its complex conjugate). Since $1 \le i \le m < n$, due to the lemma above all these sums vanish, finishing the proof. □

As an example, take $n = 4$ (square $ABCD$), P on the incircle, and $2m = 6$ (or $m = 3$; meaning we consider the sum of the sixth powers of the distances from P to A, B, C, D; Figure 3, right). Then $PA^6 + PB^6 + PC^6 + PD^6 = 252r^6$, where r denotes the inradius of the square. The result can be found easily by letting P be a midpoint of a side (e.g. of AB; $PA^6 + PB^6 + PC^6 + PD^6 = 2r^6 + 2\left(\sqrt{5}r\right)^6 = 252r^6$) and confirmed easily by dynamic geometry.

As another example, take again the square, P on the circumcircle, and sixth powers (Figure 3, right). Then $PA^6 + PB^6 + PC^6 + PD^6 = 80R^6$,

[3] If k is a multiple of n, then $z_i^k = 1$ and the sum is not 0.

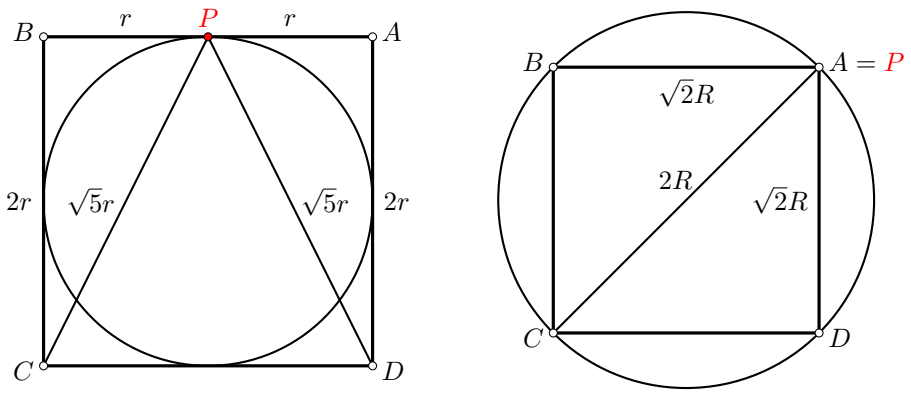


FIGURE 3. Left: Square, $P \in$ incircle. Right: Square, $P \in$ circumcircle.

which can be determined easily by letting P one of the vertices (e.g. $P = A$: $PA^6 + PB^6 + PC^6 + PD^6 = (2R)^6 + 2\left(\sqrt{2}R\right)^6 = 80R^6$) and confirmed again by using dynamic geometry.

2. Sums of Powers of Distances to *Sides* of Polygons

The constancy of the sum of squared distances to the sides of a polygon has been addressed in [7]. Exploring higher-order powers via dynamic geometry software, we observed an intriguing behavior for a regular pentagon $A_1A_2A_3A_4A_5$: the sum of fourth powers of distances from a point P to the supporting lines $a_1, \ldots, a_5$, given by $(Pa_1)^4 + \ldots + (Pa_5)^4$, remains invariant as P moves along a circle centered at the pentagon's center (see Figure 4).

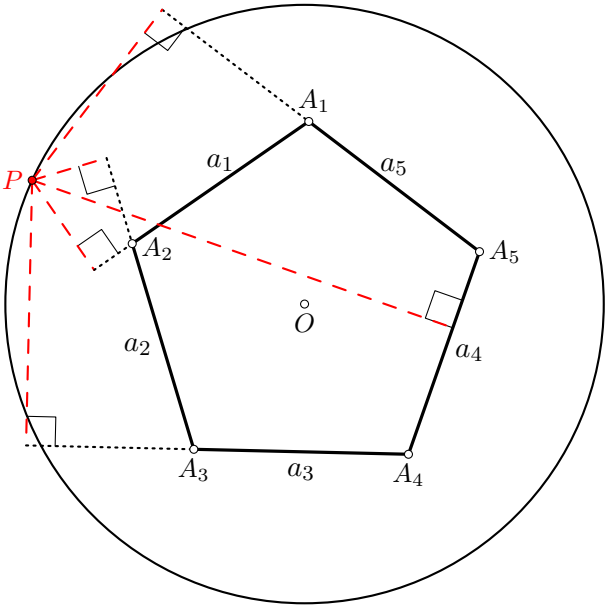


FIGURE 4. $(Pa_1)^4 + \ldots + (Pa_5)^4$ is constant.

We generalize this observation in the following theorem.

Theorem 6. *Let $a_1, \ldots, a_n$ be the sides of a regular n-gon centered at the origin, and let P be a point on a circle Γ of radius r centered at the origin. Let d_j denote the Euclidean distance from P to the supporting line of the side a_j, and let $0 \leq m < n$ be an integer.*

- *If m is even, then the sum $d_1^m + \ldots + d_n^m$ is* constant *for all points P on Γ.*
- *If m is odd, then the sum $d_1^m + \ldots + d_n^m$ is* constant *for all points P on Γ that lie inside the polygon.*

In both cases, the value of the sum is given by

$$S = n \sum_{k=0}^{m} \binom{m}{k} (-r)^k \cos^{m-k}\left(\frac{\pi}{n}\right) \frac{1}{2^k} \binom{k}{k/2}$$

where we use the convention that $\binom{u}{v} = 0$ if v is not an integer.

Proof. Without loss of generality, we may assume that the vertices of the regular n-gon lie on the unit circle and correspond to the n-th roots of unity. The distance from the origin to each side is then given by $h = \cos(\frac{\pi}{n})$ (see Figure 5). The normal vectors n_j of the supporting lines of the sides point in symmetrically distributed directions. The corresponding angles are

$$\alpha_j = \frac{2j\pi}{n} - \frac{\pi}{n} \qquad \text{for } j = 1, \ldots, n.$$

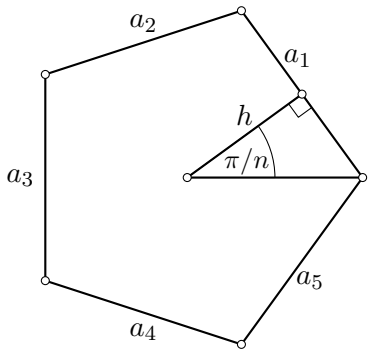


FIGURE 5. The regular n-gon with vertices on the unit circle.

The point P can be parameterized as $P = re^{i\varphi}$, where $0 \leq \varphi < 2\pi$. The Euclidean distance d_j from P to the line containing the side a_j is given by

$$d_j = |h - r\cos(\varphi - \alpha_j)| \tag{2}$$

(see Figure 6).

Observe that formula (2) holds without the absolute value if and only if P and the origin O lie on the same side of the supporting line of a_j. Consequently, the formula is valid without the absolute value for all j if and only if P lies inside the polygon. Thus, in the subsequent computations, we omit the absolute value for d_j, bearing in mind that this step is universally valid for all positions of P on Γ when m is an even power (since the sign is absorbed by the power), but requires P to lie inside the polygon if m is odd.

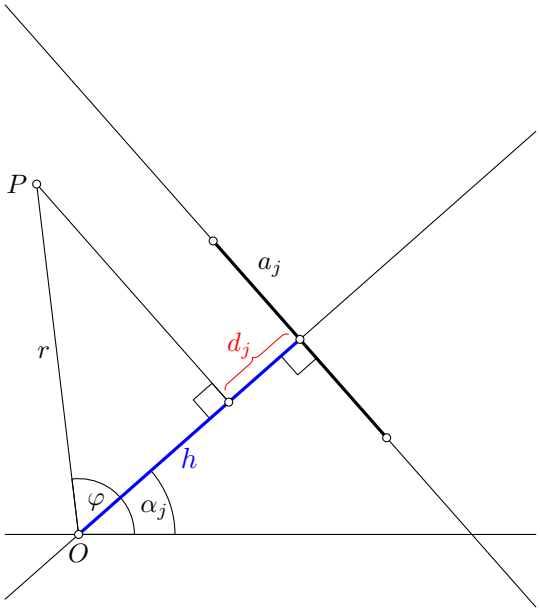


FIGURE 6. The distance d_j between P and a_j.

Taking the m^{th} power yields

$$d_j^m = (h - r\cos(\varphi - \alpha_j))^m.$$

We now consider the sum $S = \sum_{j=1}^{n} d_j^m$. Expanding each term via the binomial theorem, we obtain

$$d_j^m = \sum_{k=0}^{m} \binom{m}{k} h^{m-k} \left(-r\cos(\varphi - \alpha_j)\right)^k.$$

Interchanging the order of summation yields

$$S = \sum_{k=0}^{m} \binom{m}{k} (-r)^k h^{m-k} \sum_{j=1}^{n} \cos^k(\varphi - \alpha_j).$$

To show that S is independent of φ, it suffices to demonstrate that

$$\Sigma_k := \sum_{j=1}^{n} \cos^k(\varphi - \alpha_j)$$

is independent of φ for all integers k with $0 \le k \le m$. By Euler's formula $\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$, the k^{th} power expands to:

$$\cos^k(\theta) = \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} e^{i(k-2l)\theta}.$$

Setting $\theta = \varphi - \alpha_j$, the sum Σ_k becomes

$$\Sigma_k = \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} e^{i(k-2l)\varphi} \sum_{j=1}^{n} e^{-i(k-2l)\alpha_j}.$$

Using the relation $\alpha_j = \alpha_0 + \frac{2\pi j}{n}$ with $\alpha_0 = -\frac{\pi}{n}$, we can rewrite this as

$$\Sigma_k = \frac{1}{2^k} \sum_{l=0}^{k} \binom{k}{l} \mathrm{e}^{i(k-2l)\varphi} \mathrm{e}^{-i(k-2l)\alpha_0} \sum_{j=1}^{n} \left(\mathrm{e}^{-i(k-2l)\frac{2\pi}{n}}\right)^j .$$

We isolate the inner sum

$$W = \sum_{j=1}^{n} \left(\mathrm{e}^{-i(k-2l)\frac{2\pi}{n}}\right)^j ,$$

which represents a sum of powers of the roots of unity. We observe the following cases:

- If $q = k-2l$ is a multiple of n, the term inside the parentheses equals 1, yielding $W = n$.
- If q is not a multiple of n, the sum vanishes, so $W = 0$.

Recall that k ranges from 0 to m, and l ranges from 0 to k. Therefore, the quantity $q = k - 2l$ satisfies $-m \leq q \leq m$. Since $m < n$, it follows that $-n < q < n$. The only multiple of n within this interval is $q = 0$. This leads to the following simplification for W:

- If $k - 2l \neq 0$, then $W = 0$.
- If $k-2l = 0$ (which implies that k is even and $l = k/2$), then $W = n$.

Consequently, the sum Σ_k simplifies as follows:

- If k is odd, $k - 2l$ is never zero, hence $\Sigma_k = 0$.
- If k is even (say $k = 2\lambda$), only the term corresponding to $l = \lambda$ survives, yielding:

$$\Sigma_{2\lambda} = \frac{1}{2^{2\lambda}} \binom{2\lambda}{\lambda} \cdot 1 \cdot \mathrm{e}^0 \cdot n = \frac{n}{2^{2\lambda}} \binom{2\lambda}{\lambda}$$

Thus, the sum Σ_k, and consequently S, is indeed independent of φ. Its value is given by

$$S = n \sum_{k=0}^{m} \binom{m}{k} (-r)^k h^{m-k} \frac{1}{2^k} \binom{k}{k/2}$$

under the convention that $\binom{u}{v} = 0$ if v is not an integer. Substituting $h = \cos(\pi/n)$ yields the formula stated in the Theorem. □

3. Extension to Semi-Regular Polygons

Having established the conservation laws for distance power sums in regular polygons, we now turn our attention to semi-regular polygons—specifically semi-regular angle-$2n$-gons. A semi-regular angle-$2n$-gon possesses a natural structural decomposition: its sides alternate between two concentric, interlocking regular n-gons, G_1 and G_2, whose normal vectors are offset by an angle of π/n.

When considering the sum of the m^{th} powers of distances from a point P to the supporting lines of all $2n$ sides, $\Sigma_{n,m} := \sum_{j=1}^{2n} d_j^m$, two distinct regimes emerge:

- **The Direct Regime ($m < n$):** For exponents strictly smaller than n, the constancy of $\Sigma_{n,m}$ is a direct consequence of Theorem 6. Since each constituent regular n-gon independently yields a constant power sum for any point on the circle Γ, their combined sum is trivially constant as well.
- **The Threshold Regime ($m = n$):** The case $m = n$ represents a critical threshold. For a single regular n-gon, the n^{th} power sum does not remain constant, as it retains a residual dependence on the angular coordinate φ of P. However, when considering the full semi-regular $2n$-gon, the relative angular shift of π/n between G_1 and G_2 causes these highest-order φ-dependent terms to acquire identical magnitudes but opposite algebraic signs. As a result, a remarkable cancellation phenomenon occurs, rendering the combined sum $\Sigma_{n,n}$ invariant despite the non-constancy of its individual components.

This interaction between sub-structures and the resulting exact cancellation are formalized in the following theorem.

Theorem 7. *Given a semi-regular angle-$2n$-gon centered at the origin, let $a_1, \ldots, a_{2n}$ be its sides, and let P be a point on a circle Γ of radius r centered at the origin. Let d_j denote the Euclidean distance from P to the supporting line of the side a_j, and let $m \leq n$ be a positive integer. Let $\Sigma_{n,m} := \sum_{j=1}^{2n} d_j^m$.*

- *If m is even, then the sum $\Sigma_{n,m}$ is constant for all points P on Γ.*
- *If m is odd, then the sum $\Sigma_{n,m}$ is constant for all points P on Γ that lie in the interior of both corresponding regular n-gons.*

In both cases, the value of the sum is given by

$$S = n \sum_{k=0}^{m} \binom{m}{k} (-r)^k \frac{1}{2^k} \binom{k}{k/2} \left(h_1^{m-k} + h_2^{m-k} \right)$$

where we use the convention that $\binom{u}{v} = 0$ if v is not an integer[4].

Proof. The semi-regular angle-$2n$-gon can be split into two different regular n-gons sharing their common center. Let us denote these two regular n-gons as G_1 and G_2. The normal vectors of their supporting lines point in symmetrically distributed directions, where the normals for G_2 bisect the angles of G_1 and vice-versa.

Without loss of generality, we align the normal angles of G_1 as in the proof of Theorem 6:

$$\alpha_j = \frac{2j\pi}{n} - \frac{\pi}{n} \quad \text{for } j = 1, \ldots, n$$

Consequently, the normal angles for G_2 are shifted by $\frac{\pi}{n}$, yielding:

$$\beta_j = \alpha_j + \frac{\pi}{n} = \frac{2j\pi}{n} \quad \text{for } j = 1, \ldots, n.$$

Let h_1 and h_2 denote the respective distances from the origin to the sides of G_1 and G_2. Using the parameterization $P = re^{i\varphi}$ and omitting the absolute

[4]The meaning of h_1 and h_2 is explained in the proof.

values (which is valid for all P on Γ if m is even, and for P in the interior if m is odd), the distances to the sides of G_1 and G_2 are given by

$$d_{1,j} = h_1 - r\cos(\varphi - \alpha_j) \quad \text{and} \quad d_{2,j} = h_2 - r\cos(\varphi - \beta_j).$$

The total sum is $\Sigma_{n,m} = \sum_{j=1}^{n} d_{1,j}^m + \sum_{j=1}^{n} d_{2,j}^m$.

For $m < n$, Theorem 6 tells us that both $\sum_{j=1}^{n} d_{1,j}^m$ and $\sum_{j=1}^{n} d_{2,j}^m$ are individually independent of φ. Since the sum of two constants is constant, the statement holds trivially.

It remains to prove the theorem for the case $m = n$. Expanding the n^{th} powers via the binomial theorem, we obtain:

$$d_{1,j}^n = \sum_{k=0}^{n} \binom{n}{k} h_1^{n-k} \left(-r\cos(\varphi - \alpha_j)\right)^k$$

and analogously for $d_{2,j}^n$. Interchanging the order of summation, all terms for $k < n$ are independent of φ by the exact same mechanism established in Theorem 6. Thus, any φ-dependency in $\Sigma_{n,n}$ must originate strictly from the highest order terms where $k = n$.

For $k = n$, the coefficients h_1^{n-k} and h_2^{n-k} evaluate to $h_1^0 = h_2^0 = 1$. The sum of the $k = n$ terms from both n-gons simplifies to:

$$S_n = (-r)^n \left(\sum_{j=1}^{n} \cos^n(\varphi - \alpha_j) + \sum_{j=1}^{n} \cos^n(\varphi - \beta_j) \right).$$

Applying Euler's formula to $\cos^n(\theta)$, we expand both inner sums as in Theorem 6:

$$\sum_{j=1}^{n} \cos^n(\varphi - \alpha_j) = \frac{1}{2^n} \sum_{l=0}^{n} \binom{n}{l} \mathrm{e}^{i(n-2l)\varphi} \sum_{j=1}^{n} \mathrm{e}^{-i(n-2l)\alpha_j}.$$

Here, $q = n - 2l$ ranges from $-n$ to n. For $q = 0$ (which occurs when $l = n/2$ for even n), the term yields a constant and does not depend on φ. Unlike in the case $m < n$ of Theorem 6, the bounds $-n \le q \le n$ now allow for the non-zero multiples $q = \pm n$ (corresponding to $l = 0$ and $l = n$).

We evaluate the inner sums $W(\alpha) = \sum_{j=1}^{n} \mathrm{e}^{-i(\pm n)\alpha_j}$ as well as $W(\beta) = \sum_{j=1}^{n} \mathrm{e}^{-i(\pm n)\beta_j}$ for these boundary cases:

- For G_1, substituting $\alpha_j = \frac{2j\pi}{n} - \frac{\pi}{n}$, we have $\mathrm{e}^{-in\alpha_j} = \mathrm{e}^{-i(2j\pi - \pi)} = \mathrm{e}^{i\pi} = -1$. Therefore, $W(\alpha) = -n$.
- For G_2, substituting $\beta_j = \frac{2j\pi}{n}$, we have $\mathrm{e}^{-in\beta_j} = \mathrm{e}^{-i(2j\pi)} = 1$. Therefore, $W(\beta) = n$.

Because $W(\alpha)$ and $W(\beta)$ have the exact same magnitude but opposite signs for $q = \pm n$, their contributions perfectly cancel each other out when added together in S_n. Specifically, the φ-dependent components of the highest order term sum to:

$$\frac{(-r)^n}{2^n} \left(-n\mathrm{e}^{in\varphi} - n\mathrm{e}^{-in\varphi}\right) + \frac{(-r)^n}{2^n} \left(n\mathrm{e}^{in\varphi} + n\mathrm{e}^{-in\varphi}\right) = 0.$$

Consequently, all terms dependent on φ vanish completely. The sum $\Sigma_{n,n}$ is therefore constant.

Furthermore, we can explicitly determine the value of this constant sum $\Sigma_{n,m}$. Recall from Theorem 6 that for a single regular n-gon, where the

distance from the origin to its sides is h, the constant part of the sum is given by

$$S(h) = n \sum_{k=0}^{m} \binom{m}{k} (-r)^k h^{m-k} \frac{1}{2^k} \binom{k}{k/2}$$

where we use the convention that $\binom{k}{k/2} = 0$ if k is odd.

As established in the proof, any φ-dependent terms that arise for $m = n$ perfectly cancel each other out between the two interlocking regular n-gons G_1 and G_2. Therefore, for any $m \leq n$, the exact value of the total sum $\Sigma_{n,m}$ is strictly the addition of the constant contributions from both polygons.

Let h_1 and h_2 denote the respective distances from the origin to the sides of G_1 and G_2. The value of the sum is exactly $S(h_1) + S(h_2)$, which can be factored as:

$$\Sigma_{n,m} = n \sum_{k=0}^{m} \binom{m}{k} (-r)^k \frac{1}{2^k} \binom{k}{k/2} \left(h_1^{m-k} + h_2^{m-k} \right). \qquad \square$$

4. Signed Distances to the Sides

If we replace the Euclidean distances from P to the sides of the polygon in the Theorems 6 and 7 by signed (oriented) distances we can remove the restriction concerning the point P for odd powers. For regular polygons we obtain the following.

Corollary 8. *Let $a_1, \ldots, a_n$ be the sides of a regular n-gon centered at the origin, and let P be a point on a circle Γ of radius r centered at the origin. Let $\tilde{d}_j$ denote the oriented (signed) distance from P to the supporting line of the side a_j, defined such that the origin has a positive distance to all sides. If $0 \leq m < n$ is an integer, then the sum $\tilde{d}_1^m + \ldots + \tilde{d}_n^m$ is constant for all points P on Γ. Furthermore, the constant value of this sum is identical to the formula given in Theorem 6.*

Similarly, for semi-regular polygons we obtain the following.

Corollary 9. *Let P be a point on a circle Γ of radius r centered at the origin. Consider a semi-regular (equiangular) $2n$-gon centered at the origin, whose $2n$ supporting lines alternate in distance from the origin between h_1 and h_2. Let $\tilde{d}_j$ denote the oriented distance from P to the supporting line of the j^{th} side, chosen such that the origin has a positive distance to all sides. If m is an integer such that $0 \leq m \leq n$, then the sum $\sum_{j=1}^{2n} \tilde{d}_j^m$ is constant for all points P on Γ. Furthermore, the constant value of this sum is identical to the formula given in Theorem 7.*

The claims of both corollaries follow simply by going through the proofs of Theorems 6 and 7 and omitting the absolute value in the formula for the distance.

Conclusion. In this paper, we investigated invariant distance power sums in regular and semi-regular polygons. Beginning with an elementary observation regarding fourth powers of distances to the vertices of an equilateral triangle, we extended this invariance to arbitrary concentric circles using a complex-analytic formulation. Although the constancy for even power sums of vertex distances is established in the literature (see [10, 9]), the complex number approach presented here provides a unified, elegant framework that extends naturally to both regular and semi-regular polygons.

Our main focus then shifted to power sums of distances to the *sides* of polygons, yielding several novel insights. In contrast to vertex distance sums—which exhibit invariance exclusively for even powers—side distance sums remain constant for *odd* powers as well, provided P stays within the polygon's interior. For a regular n-gon, this invariance holds for all exponents $m < n$. Extending this analysis to semi-regular angle-$2n$-gons, we uncovered a remarkable cancellation phenomenon that preserves the sum's invariance even at the critical threshold $m = n$. Furthermore, we discussed the case of taking signed (oriented) instead of Euclidean distances, where the restriction for P to stay within the polygon's interior can be omitted.

Finally, we emphasize the indispensable role of dynamic geometry throughout this research. Interactive geometric experimentation served not only as the initial catalyst for discovering these unexpected conservation laws, but also as a powerful tool for visual verification and intuitive grounding prior to formal proof.

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