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COMMUNICATING HARMONIC PENCILS OF LINES

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Abstract. Suppose there are n harmonic pencils of lines given in the plane. We are interested in the question whether certain triples of these lines are concurrent or if triples of intersection points of these lines are collinear, provided that we impose suitable conditions on the initial harmonic pencils. Such conditions can be that certain of the given lines coincide, are concurrent or that certain intersection points are collinear. The study of these questions for $n = 2, 3, 4$ sheds light on some well known affine configurations and provides new results in the projective setting. As applications, we will formulate generalizations or stronger versions of the theorems of Pappus, Desargues, Ceva and Menelaos. Notably, the generalized theorems of Ceva and Menelaos suggest a new way to generalize the terms 'collinearity' and 'concurrency'.

1. Introduction

Before we start, let us briefly fix some notation: Points will be denoted by capital letters, lines by small letters. The intersection point of two lines g and h is $g \times h$, and $A \times B$ denotes the line trough A and B. The term $\frac{AB}{CD}$ is the quotient of the oriented lengths of the segments AB and CD on an oriented line. The cross-ratio of four collinear points A, B, C, D is

$$\mathrm{CR}(A, B; C, D) := \frac{AC}{CB} \cdot \frac{BD}{DA}.$$

A harmonic quadruple of collinear points A, B, X, Y or concurrent lines g, h, u, v will be denoted by $(AB; XY)$ and $(gh; uv)$, respectively: Y is the harmonic conjugate of X with respect to A and B (see, e.g., [1]). Finally, we denote by $[ABC]$ the signed area of a triangle ABC with respect to a given orientation of the plane. In particular $[ABC] = -[CBA]$ (see Figure 16).

Suppose we are given points $A_1, A_2, \ldots, A_n$ in the projective plane and each point A_i carries a harmonic pencil of four lines which meet in A_i. In general there will be $8n(n-1)$ further intersection points of these lines. Interesting constellations concerning these lines and their points of intersection are:

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- concurrent triples of lines,
- collinear triples of intersection points.

Of course, without any further conditions no such constellation will occur. This changes if we let the n initial harmonic pencils of lines "communicate" with each other. This means that we impose conditions of the following kind:

- certain lines coincide,
- certain lines are concurrent,
- certain points of intersection are collinear.

Let us have a look at an example with two points A_1, A_2. Here we have the following proposition, illustrated by Figure 1.

Proposition 1.1. *Let A_1, A_2 be two points in the projective plane and $(a_{i1}a_{i2}; b_{i1}b_{i2})$ harmonic pencils of lines at point A_i, $i \in \{1, 2\}$. Then the following holds: If three of the points*

$$a_{11} \times a_{21}, \quad a_{12} \times a_{22}, \quad b_{11} \times b_{21}, \quad b_{12} \times b_{22}$$

are collinear, then so are all four points.

Proof. The intersection points P, Q, X, Y of a harmonic pencil with a line g are harmonic points. And vice versa, if four collinear harmonic points are connected with a point A_1 or A_2, the connecting lines are a harmonic pencil. Therefore the claim follows from the fact that the fourth harmonic line is determined by the three others. This concludes the proof.

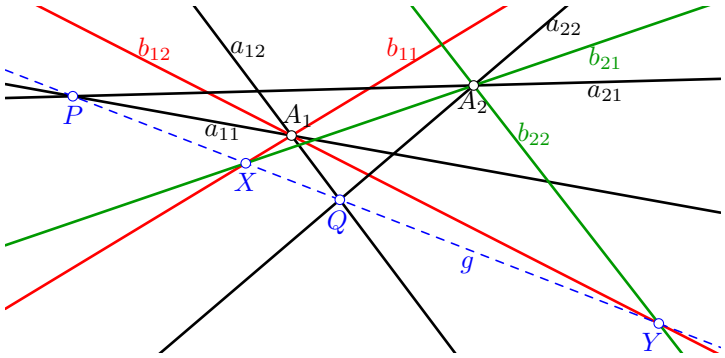

FIGURE 1. The configuration in Proposition 1.1.

A nice special case of Proposition 1.1 occurs, when $a_{11} = a_{21} = a$ (see Figure 2):

Corollary 1.1. *Let A_1, A_2 be two points in the projective plane with $a = A_1 \times A_2$, and $(a, a_i; b_{i1}b_{i2})$ harmonic pencils of lines at point A_i, $i \in \{1, 2\}$. Then the triples of points*

$$a_1 \times a_2, \quad b_{11} \times b_{21}, \quad b_{12} \times b_{22}$$
$$a_1 \times a_2, \quad b_{11} \times b_{22}, \quad b_{12} \times b_{21}$$

are collinear.

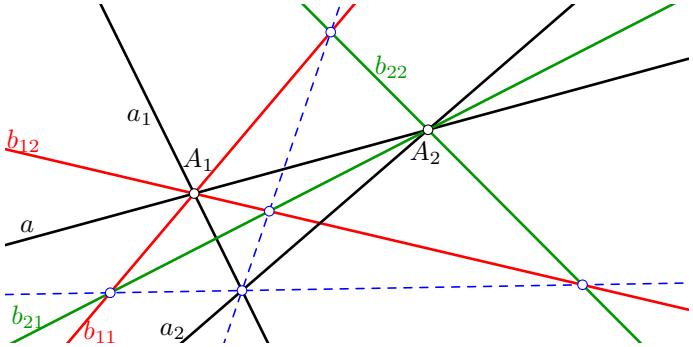

FIGURE 2. Collinear intersection points of two communicating harmonic pencils (Corollary 1.1).

In Sections 2 and 3 we will illustrate the concept of communicating harmonic pencils of lines with two examples which live in the affine plane. There the harmonic pencils do not communicate in the above described purely projective way. The idea is then to carry these two configurations into the projective plane and to replace the affine conditions by projectively invariant conditions. This will be done in Sections 4 and 5, respectively. Finally, in Section 6, we will show how our results can be used to expand some well-known theorems.

2. Harmonic pencils in affine triangles

Let $A_1A_2A_3$ be an affine triangle with sides a_1, a_2, a_3, where the notation is such that $A_i \notin a_i$. Let g_i denote the inner angle bisectors of the triangle and h_i the outer angle bisectors in A_i. Then $(a_ia_{i+1}; g_{i+2}h_{i+2})$ is a harmonic pencil of lines for $i \in \{1, 2, 3\}$ (indices are taken cyclically). Obviously, the following lines are concurrent.

(i) g_1, g_2, g_3: The intersection point is the center of the incircle of the triangle),
(ii) g_i, h_{i+1}, h_{i+2} for $i \in \{1, 2, 3\}$: The intersection points are the centers of the excircles (see Figure 3).

In Section 4 we will replace the angle bisectors in the triangle by lines g_i, h_i which form harmonic pencils together with the sides of the triangle. It will turn out that in this purely projective setting (i) occurs if and only if one of the concurrencies in (ii) occurs (see Theorem 4.1).

The configuration in the next section is less well-known.

3. Harmonic pencils in affine complete quadrilaterals

This extremely rich configuration has first been described by Jakob Steiner in [11]. We present the theorem translated from the French original, with a few additional remarks in parentheses.

Theorem 3.1 (Jakob Steiner, 1827)**.** *For a complete quadrilateral with four sides and six vertices there holds:*

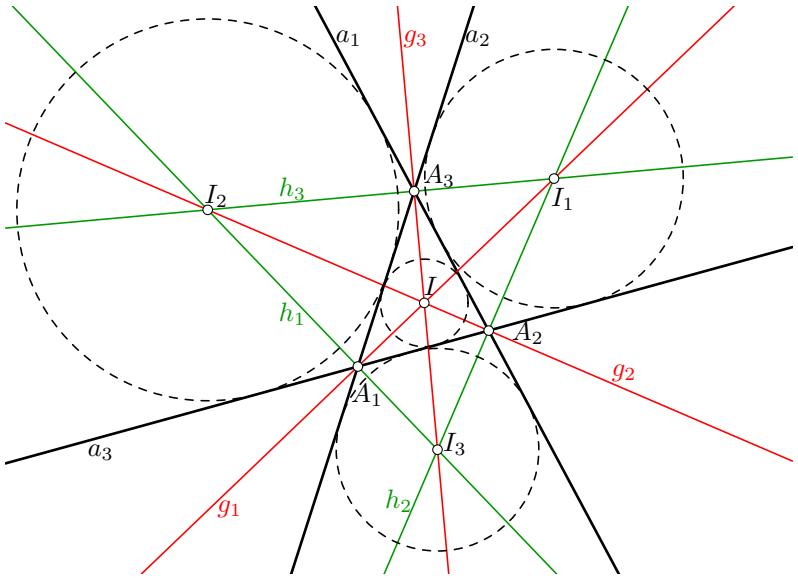

FIGURE 3. Incircle and excircles in triangle $A_1A_2A_3$.

(i) *The four sides, taken in groups of three, form four triangles. Their circumcircles pass through a single point M (the Miquel point, see* [8], [9]*).*
(ii) *The centers of these four circumcircles and M lie on a further circle.*
(iii) *The feet of the perpendiculars from M to the four sides of the quadrilateral lie on a line s (the Simson-Wallace line), and M is the only point with this property.*
(iv) *The orthocenters of the four triangles in (i) lie on a line c. (c is the common radical axis of the three Thales circles over the diagonals of the complete quadrilateral: Bodenmiller-Steiner Theorem, see* [6, p. 138]*).*
(v) *The lines s and c are parallel. c is the middle line between M and s.*
(vi) *The midpoints of the three diagonals of the complete quadrilateral lie on a line n (Gauß-Newton Theorem, see* [10], [2], [3, p. 112–121]*).*
(vii) *The line n is perpendicular to c and s.*
(viii) *Each of the four triangles of the quadrilateral has an incircle and three excircles which gives a total of sixteen circles. The centers C_k of these circles lie in groups of four on eight new circles.*
(ix) *The eight circles mentioned in (viii) can be decomposed into two groups of four circles. The four circles in on group are orthogonal to the four circles of the other group. In particular, the four centers of the circles of both groups lie each on a line, and these two lines are orthogonal. (And the four circles in one of the two groups meet in two points on the line which carries the centers of the other group.)*
(x) *The two perpendicular lines mentioned in (ix) meet in M.*

We add the following observations to complete the picture and to focus on the part of the constellation which is relevant for our purposes:

Theorem 3.2. *Let A_1, A_2, A_3, A_4 be the vertices of a quadrilateral with sides $a_{12} = A_1 \times A_2$, $a_{23} = A_2 \times A_3$, $a_{34} = A_3 \times A_4$, $a_{41} = A_4 \times A_1$, and let $A_5 = a_{12} \times a_{34}$, $A_6 = a_{23} \times a_{41}$. Denote by g_i and h_i the inner and outer angle bisectors in the points A_i, $i \in \{1, \dots, 6\}$ (see Figure 4). Then there holds:*

(xi) *The following quintuples of points are collinear:*

$$\begin{array}{llll} A_5, & h_1 \times h_4, & g_1 \times g_4, & h_3 \times h_2, \quad g_3 \times g_2 \text{ (on } g_5) \\ A_6, & h_3 \times h_4, & g_3 \times g_4, & h_1 \times h_2, \quad g_1 \times g_2 \text{ (on } g_6) \\ A_5, & g_1 \times h_4, & h_1 \times g_4, & g_3 \times h_2, \quad h_3 \times g_2 \text{ (on } h_5) \\ A_6, & g_3 \times h_4, & h_3 \times g_4, & g_1 \times h_2, \quad h_1 \times g_2 \text{ (on } h_6) \end{array}$$

Stated differently: The sixteen centers C_k of the circles mentioned in (viii) lie in groups of four on the angle bisectors of the complete quadrilateral in the points $A_1, A_2, \dots, A_6$: In each center C_k, three of these angle bisectors meet.

(xii) *If A_i, A_j lie on a side of the complete quadrilateral, then there are four of the centers C_k mentioned in (viii), say $C_{k_1}, C_{k_2}, C_{k_3}, C_{k_4}$, such that $A_i, A_j, C_{k_1}, C_{k_2}$ and $A_i, A_j, C_{k_3}, C_{k_4}$ each lie on a circle. This gives rise to 24 new circles: In each center C_k, three of theses circles meet, together with two of the eight circles mentioned in (viii).*

(xiii) *Each angle bisector carries two centers of the 24 new circles mentioned in (xii).*

Proof.

(xi) It is clear from the construction, that the centers C_k of the incircles and excircles lie on the angle bisectors in the points $A_1, A_2, \dots, A_6$: See Figure 4.

(xii),(xiii) Since the two angle bisectors in each of the points $A_1, \dots, A_6$ are orthogonal, the 24 new circles are just Thales circles over two centers. See Figure 5, where the situation is illustrated for the segment A_2, A_5 and two centers C and D. This completes the proof.

Tow sides of the quadrilateral meeting in A_i form together with the angle bisectors in this point a harmonic pencil of lines. According to Theorem 3.2 (xi), groups of four of the intersection points of angle bisectors and one of the points A_i are collinear. In Section 5 we will consider this situation where we replace the angle bisectors by harmonic lines.

4. Harmonic pencils in projective triangles

We now replace the angle bisectors in a triangle by lines such that in each vertex a harmonic pencil of lines results. Each pair of these three harmonic pencils shares a common line (a side of the triangle), and they are therefore not completely independent. And indeed, certain intersection points of lines turn out to be collinear.

Proposition 4.1. *Let A_1, A_2, A_3 be a triangle with sides a_i, $i \in \{1, 2, 3\}$ and such that $A_i \notin a_1$. Through each vertex A_i we choose two lines g_i, h_i*

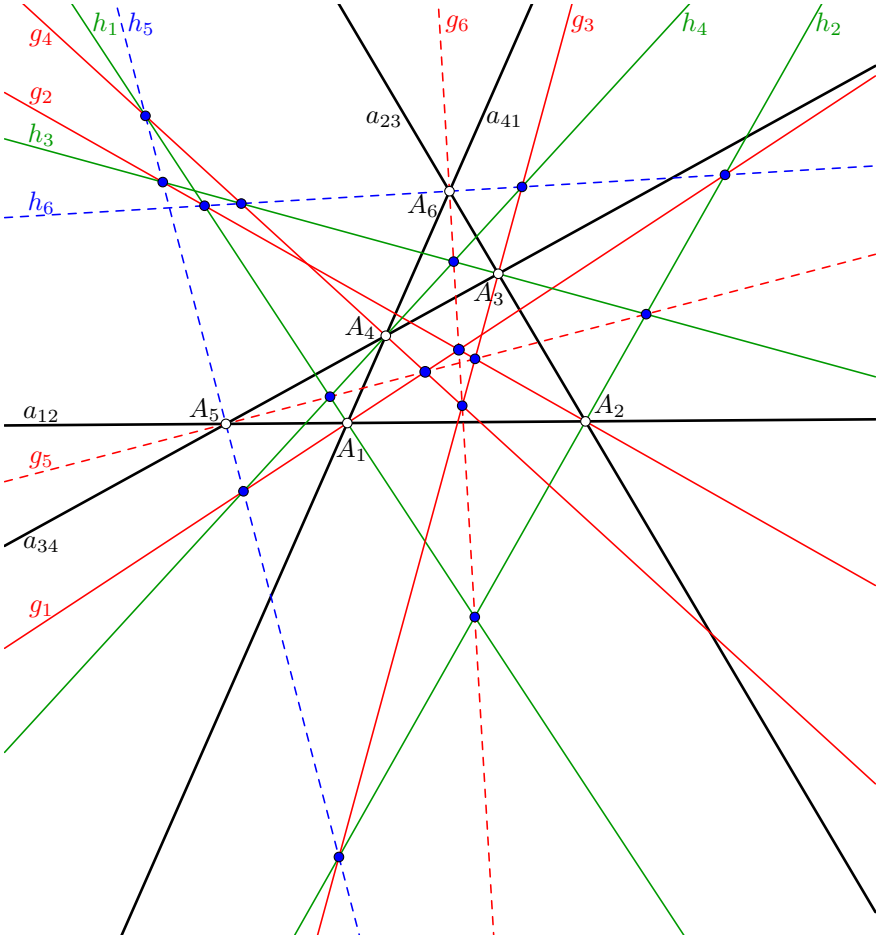

FIGURE 4. Complete quadrilateral with angle bisectors.

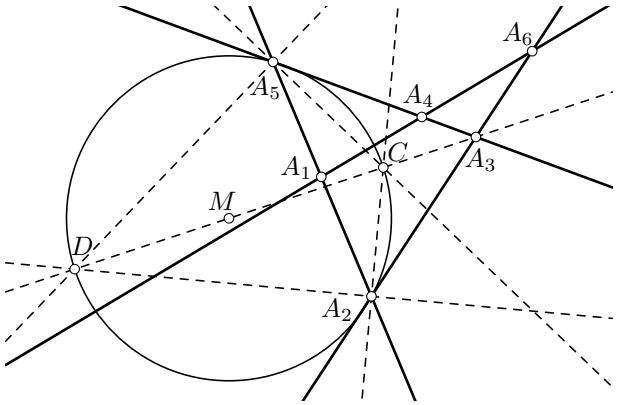

FIGURE 5. The center C of the incircle of the triangle $A_2A_3A_5$, the center D of the excircle opposite of its vertex A_3, and the vertices A_2, A_5 lie on a circle whose center M is the midpoint of the segment CD.

such that $(a_ia_{i+1}; g_{i+2}h_{i+2})$ *is a harmonic pencil of lines (indices are taken cyclically). Then, there holds:*

(i) *The following triples of points are collinear:*

$$\begin{aligned} &\text{on line } u_i: && A_i, && g_{i+1} \times g_{i+2}, && h_{i+1} \times h_{i+2} \\ &\text{on line } v_i: && A_i, && g_{i+1} \times h_{i+2}, && g_{i+2} \times h_{i+1} \end{aligned}$$

for $i \in \{1, 2, 3\}$ *(see the dashed lines in Figure 6).*

(ii) *The lines* $(a_ia_{i+1}; u_{i+2}v_{i+2})$ *are harmonic pencils.*

Proof. (i): This follows directly by applying Corollary 1.1 to the three pairs of points A_i, A_j, $i \neq j$.

(ii): The harmonic pencil $(a_1a_3; h_2g_2)$ cuts the line h_3 in harmonic points, i.e., the points

$$(a_1 \times h_3, a_3 \times h_3; h_2 \times h_3, g_2 \times h_3) = (a_2 \times h_3, a_3 \times h_3; u_1 \times h_3, v_1 \times h_3)$$

are harmonic. But this implies that $(a_2a_3; u_1v_1)$ is a harmonic pencil. The remaining cases are analogous.

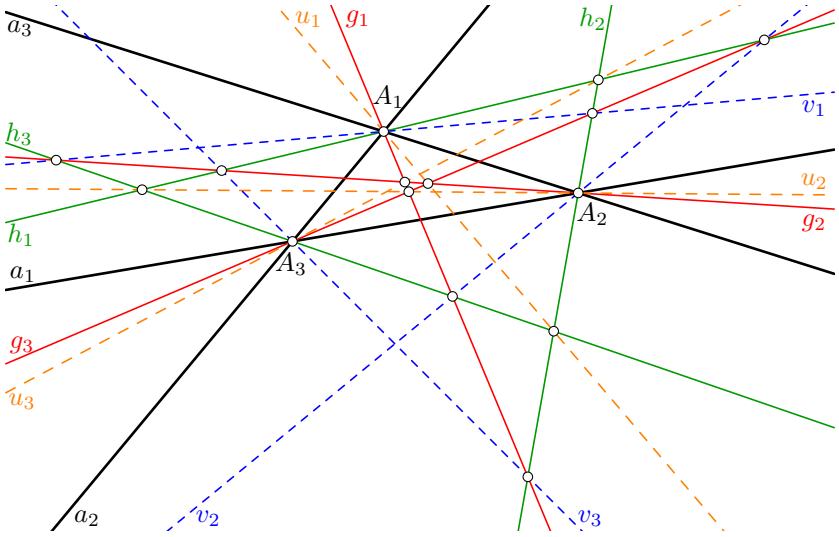

FIGURE 6. Harmonic pencils in a triangle.

If we compare the situation in Section 2 of a triangle with angle bisectors to the Proposition 4.1 above, we observe that here neither the lines g_1, g_2, g_3 nor the lines g_i, h_{i+1}, h_{i+2} are concurrent in general. However, the next theorem shows that one such concurrency already implies all other possible concurrencies.

Theorem 4.1. *Let* A_1, A_2, A_3 *be a triangle with sides* a_i, $i \in \{1, 2, 3\}$ *and such that* $A_i \notin a_1$. *Through each vertex* A_i *we choose two lines* g_i, h_i *such that* $(a_ia_{i+1}; g_{i+2}h_{i+2})$ *is a harmonic pencil of lines (indices are taken cyclically). Then we have:*

(i) *If the lines* g_1, g_2, g_3 *are concurrent, then so are the triples of lines* g_i, h_{i+1}, h_{i+2} *for* $i \in \{1, 2, 3\}$.

(ii) *Vice versa, if one of the triples g_i, h_{i+1}, h_{i+2} is concurrent, then so are the other two triples and the triple g_1, g_2, g_3.*

(iii) *Each of the concurrencies in (i) and (ii) occurs if and only if*

$$\frac{A_1B_3}{B_3A_2} \cdot \frac{A_2B_1}{B_1A_3} \cdot \frac{A_3B_2}{B_2A_1} = 1$$

where $B_i = a_i \times g_i$ (see Figure 7).

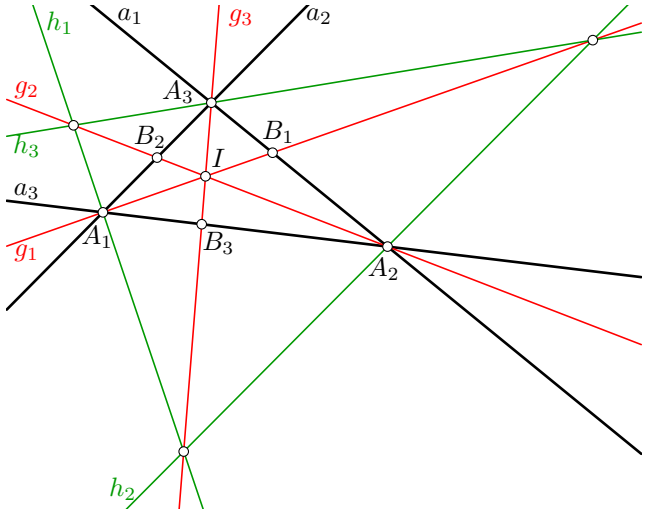

FIGURE 7. Harmonic pencils in the vertices of a triangle.

Proof. (i) Let I be the intersection point of g_1, g_2, g_3, $I_{12} = g_1 \times h_2$ and $I_{13} = g_1 \times h_3$. We want to show that $I_{12} = I_{13}$. Since $(a_ia_{i+1}; g_{i+2}h_{i+2})$ are harmonic pencils for $i \in \{2, 3\}$, we have

$$\begin{aligned} \frac{A_1I}{IB_1} &= -\frac{A_1I_{12}}{I_{12}B_1} \\ \frac{A_1I}{IB_1} &= -\frac{A_1I_{13}}{I_{13}B_1}. \end{aligned}$$

Hence $\frac{A_1I_{12}}{I_{12}B_1} = \frac{A_1I_{13}}{I_{13}B_1}$ and therefore $I_{12} = I_{13}$. Thus, g_1, h_2, h_3 are concurrent. The remaining cases are analogous.

(ii) This follows from (i) by exchanging the role of g_i and h_i in two of the vertices A_i.

(iii) This is Ceva's Theorem.

5. Harmonic pencils in projective complete quadrilaterals

We consider a quadrilateral like in Section 3 and replace the angle bisectors by lines which form together with the sides of the quadrilateral harmonic pencils. Since neighboring pencils share a common side they are not completely independent, and certain intersection points of the lines turn out to be collinear:

Proposition 5.1. *Let A_1, A_2, A_3, A_4 be the vertices of a quadrilateral with sides $a_{i(i+1)} = A_i \times A_{i+1}$ for $i \in \{1,2,3,4\}$ (where indices are taken cyclically). The quadrilateral is completed with $A_5 = a_{12} \times a_{34}$, $A_6 = a_{41} \times a_{23}$. We assume that in each vertex A_1, A_2, A_3, A_4 two lines g_i, h_i build together with the sides a harmonic pencil $(a_{(i-1)i} a_{i(i+1)}; g_i h_i)$. Then the following triples of points are collinear:*

$$
\begin{array}{llll}
A_5, & g_1 \times h_4, & h_1 \times g_4, & \qquad A_6, \quad g_1 \times h_2, \quad h_1 \times g_2, \\
A_5, & g_3 \times h_2, & h_3 \times g_2, & \qquad A_6, \quad g_3 \times h_4, \quad h_3 \times g_4, \\
A_5, & g_1 \times g_4, & h_1 \times h_4, & \qquad A_6, \quad g_1 \times g_2, \quad h_1 \times h_2, \\
A_5, & g_2 \times g_3, & h_2 \times h_3, & \qquad A_6, \quad g_3 \times g_4, \quad h_3 \times h_4
\end{array}
$$

(see the dashed lines in Figure 8).

Proof. This follows directly by applying Corollary 1.1 to neighboring pairs of vertices of the quarilateral A_1, A_2, A_3, A_4.

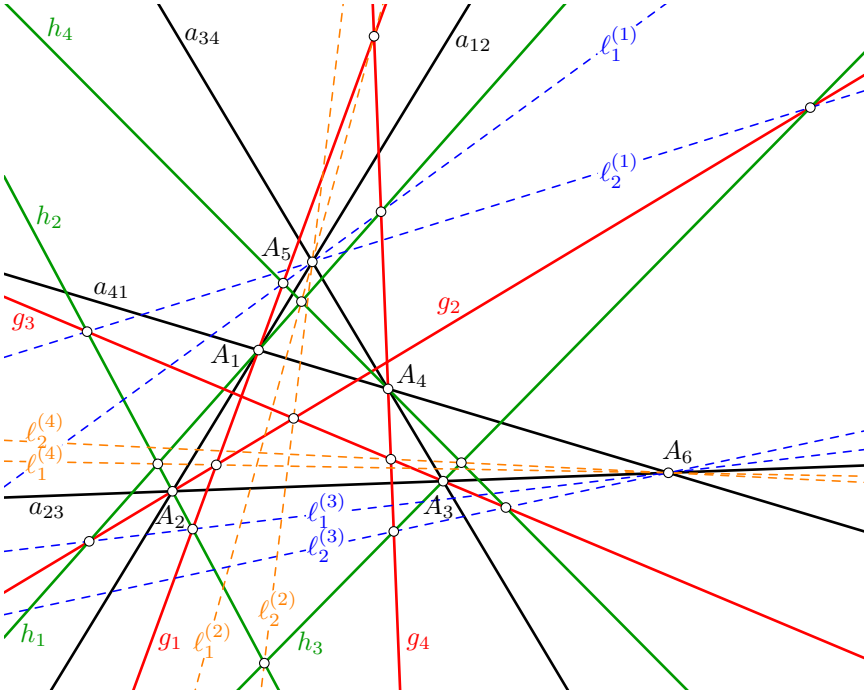

FIGURE 8. Harmonic pencils in a quadrilateral.

Compared to the situation in Theorem 3.2, we observe that the four lines which carry the quintuples of collinear intersection points each split up in two lines (compare the dashed lines in Figures 4 and 8). One can move e.g. g_1 in Proposition 5.1 such that two of the split lines collapse. Then, interestingly, not only one of the pairs of lines collapses, but all four pairs collapse simultaneously:

Theorem 5.1. *Let A_1, A_2, A_3, A_4 be the vertices of a quadrilateral with sides $a_{i(i+1)} = A_i \times A_{i+1}$ for $i \in \{1,2,3,4\}$ (where indices are taken cyclically).*

The quadrilateral is completed with $A_5 = a_{12} \times a_{34}$, $A_6 = a_{41} \times a_{23}$. Assume that in each vertex A_1, A_2, A_3, A_4 two lines g_i, h_i build together with the sides a harmonic pencil $(a_{(i-1)i}a_{i(i+1)}; g_i h_i)$. Finally, denote by $\ell^{(i)}$ the following four pairs of lines (see the dashed lines in Figure 8):

$$\begin{aligned}
&\text{line } \ell_1^{(1)}\text{: } A_5,\ g_1 \times h_4,\ h_1 \times g_4 \quad \text{and} \quad \text{line } \ell_2^{(1)}\text{: } A_5,\ g_3 \times h_2,\ h_3 \times g_2\\
&\text{line } \ell_1^{(2)}\text{: } A_5,\ g_1 \times g_4,\ h_1 \times h_4 \quad \text{and} \quad \text{line } \ell_2^{(2)}\text{: } A_5,\ g_2 \times g_3,\ h_2 \times h_3\\
&\text{line } \ell_1^{(3)}\text{: } A_6,\ g_1 \times h_2,\ h_1 \times g_2 \quad \text{and} \quad \text{line } \ell_2^{(3)}\text{: } A_6,\ g_3 \times h_4,\ h_3 \times g_4\\
&\text{line } \ell_1^{(4)}\text{: } A_6,\ g_1 \times g_2,\ h_1 \times h_2 \quad \text{and} \quad \text{line } \ell_2^{(4)}\text{: } A_6,\ g_3 \times g_4,\ h_3 \times h_4
\end{aligned}$$

Then, if the two lines of one of the pairs $\ell^{(i)}$ coincide, then the two lines of all pairs coincide, and this is the case if and only if one of the following two equivalent conditions holds:

(i) $$\prod_{i=1}^{4} \frac{A_i B_{i(i+1)(i+2)}}{B_{i(i+1)(i+2)} A_{i+1}} \cdot \frac{A_i B_{i(i+1)(i+3)}}{B_{i(i+1)(i+3)} A_{i+1}} = 1$$
where $B_{ijk} = a_{ij} \times g_k$ (see Figure 10).

(ii) $$\frac{A_1 D_2}{D_2 A_3} \cdot \frac{A_2 D_3}{D_3 A_4} \cdot \frac{A_3 D_4}{D_4 A_1} \cdot \frac{A_4 D_1}{D_1 A_2} = 1$$
where $D_1 = g_1 \times (A_2 \times A_4)$, $D_3 = g_3 \times (A_2 \times A_4)$, $D_2 = g_2 \times (A_1 \times A_3)$, $D_4 = g_4 \times (A_1 \times A_3)$ (see Figure 11).

Proof. According to Proposition 4.1(ii), the lines $(a_{12}a_{34}; \ell_1^{(2)}\ell_1^{(1)})$ and $(a_{12}a_{34}; \ell_2^{(2)}\ell_2^{(1)})$ are harmonic. Therefore, we have $\ell_1^{(1)} = \ell_2^{(1)}$ if and only if $\ell_1^{(2)} = \ell_2^{(2)}$. The same argument yields that $\ell_1^{(3)} = \ell_2^{(3)}$ if and only if $\ell_1^{(4)} = \ell_2^{(4)}$.

Suppose now that $\ell_1^{(2)} = \ell_2^{(2)}$. In order to show that this implies $\ell_1^{(4)} = \ell_2^{(4)}$ we use the Theorem of Menelaos for the following triangles and transversal lines:

- triangle $A_1A_4A_5$ with line a_{23}:

$$\frac{A_4A_6}{A_6A_1} \cdot \frac{A_1A_2}{A_2A_5} \cdot \frac{A_5A_3}{A_3A_4} = -1. \tag{1}$$

- triangle $A_1A_5(g_1 \times g_4)$ with line g_2:

$$\frac{A_1(g_1 \times g_2)}{(g_1 \times g_2)(g_1 \times g_4)} \cdot \frac{(g_1 \times g_4)(g_2 \times g_3)}{(g_2 \times g_3)A_5} \cdot \frac{A_5A_2}{A_2A_1} = -1. \tag{2}$$

- triangle $A_4A_5(g_1 \times g_4)$ with line g_3:

$$\frac{(g_1 \times g_4)(g_3 \times g_4)}{(g_3 \times g_4)A_4} \cdot \frac{A_4A_3}{A_3A_5} \cdot \frac{A_5(g_2 \times g_3)}{(g_2 \times g_3)(g_1 \times g_4)} = -1. \tag{3}$$

Multiplication of the equations (1)–(3) yields

$$\frac{A_1(g_1 \times g_2)}{(g_1 \times g_2)(g_1 \times g_4)} \cdot \frac{(g_1 \times g_4)(g_3 \times g_4)}{(g_3 \times g_4)A_4} \cdot \frac{A_4A_6}{A_6A_1} = -1. \tag{4}$$

Using Menelaos and (4) in the triangle $A_1(g_1 \times g_4)A_4$ yields that the points $g_1 \times g_2$, $g_3 \times g_4$, and A_6 are collinear. Hence we have $\ell_1^{(4)} = \ell_2^{(4)}$ as claimed. The same argument shows that $\ell_1^{(2)} = \ell_2^{(2)}$ follows from $\ell_1^{(4)} = \ell_2^{(4)}$.

Now we show that $\ell_1^{(2)} = \ell_2^{(2)}$ is equivalent to condition (i) in the theorem. Before we start, we set up some identities which hold in general, i.e., without the hypothesis $\ell_1^{(2)} = \ell_2^{(2)}$. To this end, we apply Ceva's Theorem to the following triangles:

- triangle $A_1A_5A_4$ with point $g_1 \times g_4$:

$$\frac{A_1B_{124}}{B_{124}A_5} \cdot \frac{A_5B_{341}}{B_{341}A_4} \cdot \frac{A_4C_{41}}{C_{41}A_1} = 1, \tag{5}$$

  where $C_{41} = a_{41} \times \ell_1^{(2)}$.
- triangle $A_2A_5A_3$ with point $g_2 \times g_3$:

$$\frac{A_5B_{123}}{B_{123}A_2} \cdot \frac{A_2C_{23}}{C_{23}A_3} \cdot \frac{A_3B_{342}}{B_{342}A_5} = 1, \tag{6}$$

  where $C_{23} = a_{23} \times \ell_2^{(2)}$.

Now we employ Menelaos once more as follows:

- triangle $A_1A_5A_4$ with transversal line g_2:

$$\frac{A_1A_2}{A_2A_5} \cdot \frac{A_5B_{342}}{B_{342}A_4} \cdot \frac{A_4B_{412}}{B_{412}A_1} = -1. \tag{7}$$

- triangle $A_1A_5A_4$ with transversal line g_3:

$$\frac{A_1B_{123}}{B_{123}A_5} \cdot \frac{A_5A_3}{A_3A_4} \cdot \frac{A_4B_{413}}{B_{413}A_1} = -1. \tag{8}$$

- triangle $A_2A_5A_3$ with transversal line g_1:

$$\frac{A_5A_1}{A_1A_2} \cdot \frac{A_2B_{231}}{B_{231}A_3} \cdot \frac{A_3B_{341}}{B_{341}A_5} = -1. \tag{9}$$

- triangle $A_2A_5A_3$ with transversal line g_4:

$$\frac{A_5B_{124}}{B_{124}A_2} \cdot \frac{A_2B_{234}}{B_{234}A_3} \cdot \frac{A_3A_4}{A_4A_5} = -1. \tag{10}$$

The product of the six equations (5)–(10) simplifies to

$$\zeta \cdot \frac{A_1A_5}{A_5A_2} \cdot \frac{A_2C_{23}}{C_{23}A_3} \cdot \frac{A_3A_5}{A_5A_4} \cdot \frac{A_4C_{41}}{C_{41}A_1} = 1,$$

where ζ is the expression (i) in the theorem. Hence, $\zeta = 1$ if and only if

$$\frac{A_1A_5}{A_5A_2} \cdot \frac{A_2C_{23}}{C_{23}A_3} \cdot \frac{A_3A_5}{A_5A_4} \cdot \frac{A_4C_{41}}{C_{41}A_1} = 1. \tag{11}$$

So, we are done if we can show that (11) is equivalent to the condition $\ell_1^{(2)} = \ell_2^{(2)}$. Indeed, define the points $C'_{23} = \ell_2^{(2)} \times (A_2 \times A_4)$ and $C'_{41} = \ell_1^{(2)} \times (A_2 \times A_4)$ (see Figure 9). We use again the Theorem of Menelaos as follows:

- triangle $A_1A_2A_4$ with transversal line $\ell_1^{(2)}$:

$$\frac{A_1A_5}{A_5A_2} \cdot \frac{A_2C'_{41}}{C'_{41}A_4} \cdot \frac{A_4C_{41}}{C_{41}A_1} = -1. \tag{12}$$

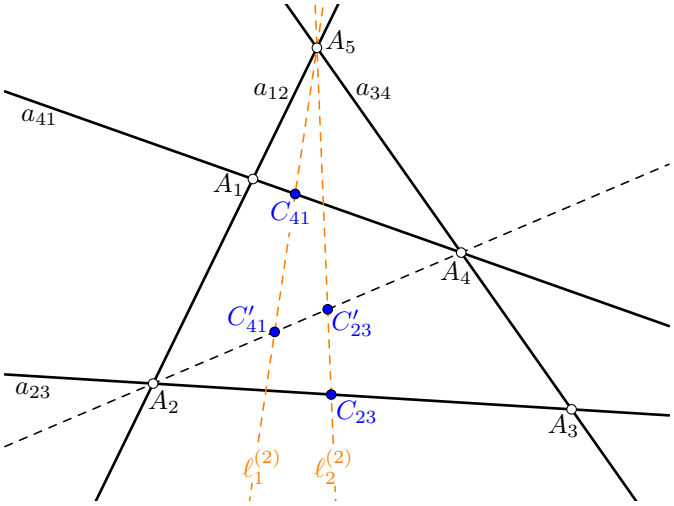

FIGURE 9. Equivalence of (11) and $\ell_1^{(2)} = \ell_2^{(2)}$.

- triangle $A_2A_3A_4$ with transversal line $\ell_2^{(2)}$:

$$\frac{A_2C_{23}}{C_{23}A_3} \cdot \frac{A_3A_5}{A_5A_4} \cdot \frac{A_4C'_{23}}{C'_{23}A_2} = -1. \tag{13}$$

The product of (12) and (13) is

$$\frac{A_1A_5}{A_5A_2} \cdot \frac{A_2C_{23}}{C_{23}A_3} \cdot \frac{A_3A_5}{A_5A_4} \cdot \frac{A_4C_{41}}{C_{41}A_1} \cdot \frac{A_2C'_{41}}{C'_{41}A_4} \cdot \frac{A_4C'_{23}}{C'_{23}A_2} = 1.$$

Hence, equation (11) holds if and only if

$$\frac{A_2C'_{41}}{C'_{41}A_4} \cdot \frac{A_4C'_{23}}{C'_{23}A_2} = 1,$$

or equivalently

$$\frac{A_2C'_{41}}{C'_{41}A_4} = \frac{A_2C'_{23}}{C'_{23}A_4}.$$

But this is indeed true if and only if $\ell_1^{(2)} = \ell_2^{(2)}$, as claimed.

It remains to show that the conditions (i) and (ii) in the theorem are equivalent. We use again Menelaos (see Figure 11):

- triangle $A_4A_2A_3$ with transversal line g_1:

$$\frac{A_4D_1}{D_1A_2} \cdot \frac{A_2B_{231}}{B_{231}A_3} \cdot \frac{A_3B_{341}}{B_{341}A_4} = -1. \tag{14}$$

- triangle $A_1A_3A_4$ with transversal line g_2:

$$\frac{A_1D_2}{D_2A_3} \cdot \frac{A_3B_{342}}{B_{342}A_4} \cdot \frac{A_4B_{412}}{B_{412}A_1} = -1. \tag{15}$$

- triangle $A_2A_4A_1$ with transversal line g_3:

$$\frac{A_2D_3}{D_3A_4} \cdot \frac{A_4B_{413}}{B_{413}A_1} \cdot \frac{A_1B_{123}}{B_{123}A_2} = -1. \tag{16}$$

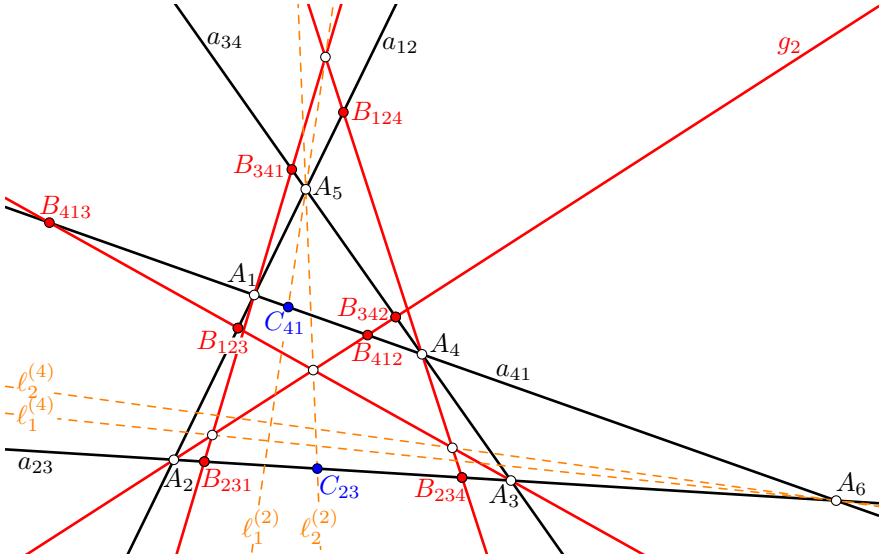

FIGURE 10. Lines and points in Theorem 5.1(i).

- triangle $A_3A_1A_2$ with transversal line g_4:

$$\frac{A_3D_4}{D_4A_1}\cdot\frac{A_1B_{124}}{B_{124}A_2}\cdot\frac{A_2B_{234}}{B_{234}A_3}=-1. \tag{17}$$

The product of (14)–(17) is

$$\zeta\cdot\frac{A_4D_1}{D_1A_2}\cdot\frac{A_1D_2}{D_2A_3}\cdot\frac{A_2D_3}{D_3A_4}\cdot\frac{A_3D_4}{D_4A_1}=1$$

which shows that indeed (i) and (ii) in the theorem are equivalent.

Remark. Note that if we fix three of the four lines $g_1,\dots,g_4$, then there exists a unique position for the fourth line such that the criterion (ii) in Theorem 5.1 is satisfied.

6. Broadening classical results

Condition (ii) in Theorem 5.1 is particularly interesting as it can be framed in various ways that give further insights to specific constellations. First, we can derive a statement about quadruples of points with the same cross-ratio:

Corollary 6.1. *Let A_1,A_2,A_3,A_4 and B_1,B_2,B_3,B_4 be two quadruples of collinear points. Then the following statements are equivalent:*

(i) $(A_1\times B_1)\times(A_2\times B_2),(A_3\times B_3)\times(A_4\times B_4),(A_1\times B_3)\times(A_4\times B_2),(A_1\times B_4)\times(A_3\times B_2),(A_2\times B_3)\times(A_4\times B_1),(A_2\times B_4)\times(A_3\times B_1)$ *are collinear.*

(ii) $(A_1\times B_1)\times(A_3\times B_3),(A_2\times B_2)\times(A_4\times B_4),(A_1\times B_2)\times(A_4\times B_3),(A_1\times B_4)\times(A_2\times B_3),(A_2\times B_1)\times(A_3\times B_4),(A_3\times B_2)\times(A_4\times B_1)$ *are collinear.*

(iii) $(A_1\times B_1)\times(A_4\times B_4),(A_2\times B_2)\times(A_3\times B_3),(A_1\times B_2)\times(A_3\times B_4),(A_1\times B_3)\times(A_2\times B_4),(A_2\times B_1)\times(A_4\times B_3),(A_3\times B_1)\times(A_4\times B_2)$ *are collinear.*

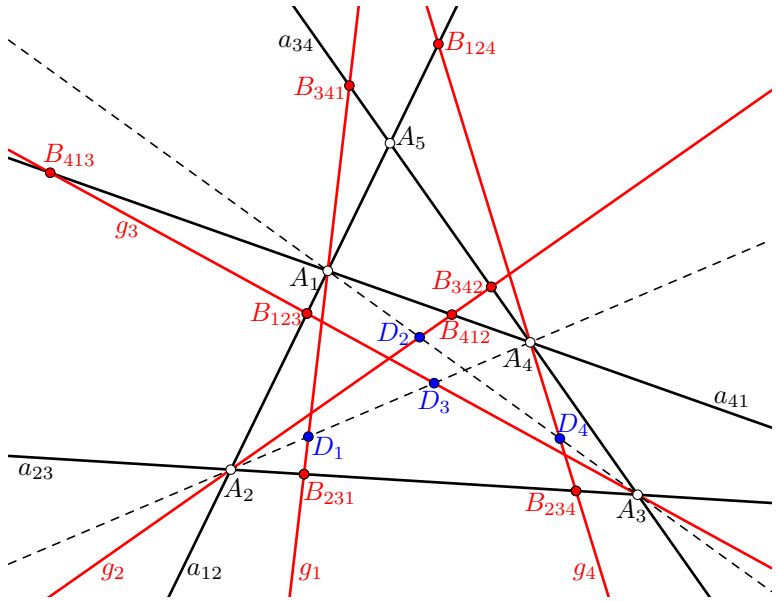

FIGURE 11. Lines and points in Theorem 5.1(ii).

(iv) $(A_1 \times B_2) \times (A_2 \times B_1), (A_2 \times B_3) \times (A_3 \times B_2), (A_3 \times B_4) \times (A_4 \times B_3), (A_4 \times B_1) \times (A_1 \times B_4), (A_1 \times B_3) \times (A_3 \times B_1), (A_2 \times B_4) \times (A_4 \times B_2)$ *are collinear.*

(v) $\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_1, B_2; B_3, B_4)$

Proof. By applying Theorem 5.1 (ii) to the quadrilateral $A_1B_1A_2B_2$ and the lines $g_1 = A_1 \times B_3$, $g_2 = B_1 \times A_3$, $g_3 = A_2 \times B_4$, $g_4 = B_2 \times A_4$ we get that

$$\frac{A_1A_3}{A_3A_2} \cdot \frac{B_1B_4}{B_4B_2} \cdot \frac{A_2A_4}{A_4A_1} \cdot \frac{B_2B_3}{B_3B_1} = 1, \tag{18}$$

or equivalently, $\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_1, B_2; B_3, B_4)$, is equivalent to the collinearity of the points

$$(A_1 \times B_1) \times (A_2 \times B_2), \quad (A_1 \times B_3) \times (A_4 \times B_2), \quad (A_2 \times B_4) \times (A_3 \times B_1)$$

(see pair $\ell^{(2)}$ in Theorem 5.1).

Analogously, $\mathrm{CR}(A_3, A_4; A_1, A_2) = \mathrm{CR}(B_3, B_4; B_1, B_2)$ is equivalent to the collinearity of the points

$$(A_3 \times B_3) \times (A_4 \times B_4), \quad (A_3 \times B_1) \times (A_2 \times B_4), \quad (A_4 \times B_2) \times (A_1 \times B_3),$$

$\mathrm{CR}(A_1, A_2; A_4, A_3) = \mathrm{CR}(B_1, B_2; B_4, B_3)$ is equivalent to the collinearity of the points

$$(A_1 \times B_1) \times (A_2 \times B_2), \quad (A_1 \times B_4) \times (A_3 \times B_2), \quad (A_2 \times B_3) \times (A_4 \times B_1)$$

and $\mathrm{CR}(A_3, A_4; A_2, A_1) = \mathrm{CR}(B_3, B_4; B_2, B_1)$ is equivalent to the collinearity of the points

$$(A_3 \times B_3) \times (A_4 \times B_4), \quad (A_3 \times B_2) \times (A_1 \times B_4), \quad (A_4 \times B_1) \times (A_2 \times B_3).$$

Since all the cross-ratio equations are in fact equivalent and because the triples of points have sufficiently many common points, it follows that

$$\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_1, B_2; B_3, B_4) \tag{19}$$

is equivalent to the collinearity of all those points, which proves the equivalence of (i) and (v).

Now (ii), (iii) and (iv) follow by noting that

$$\mathrm{CR}(A_1, A_3; A_2, A_4) = \mathrm{CR}(B_1, B_3; B_2, B_4), \tag{20}$$

$$\mathrm{CR}(A_1, A_4; A_2, A_3) = \mathrm{CR}(B_1, B_4; B_2, B_3) \tag{21}$$

and

$$\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_2, B_1; B_4, B_3) \tag{22}$$

are all equivalent to (19) and taking the corresponding intersection points. This concludes the proof.

Note that the ordinary Pappus Theorem follows from Corollary 6.1 by letting A_1 and A_3 as well as B_1 and B_3 coincide: This implies immediately that $\mathrm{CR}(A_1, A_2; A_3, A_4) = 0 = \mathrm{CR}(B_1, B_2; B_3, B_4)$ hence (iv) in Corollary 6.1 will always be true regardless of the constellation of A_1, A_2, A_4, B_1, B_2, B_4.

Interestingly, Corollary 6.1 (iv) is a particularly nice generalization of Pappus' Theorem with four points on each of two lines:

Corollary 6.2. *Let $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ be collinear points, respectively. By avoiding two of the points A_i, B_i, the remaining six points form a hexagon H_i with sides $A_j \times B_k$, $i \neq j \neq k \neq i$. The intersections of opposite sides of the hexagon H_i lie on a Pappus line p_i. Then, two of these Pappus lines (and hence all four) coincide if and only if* $\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_1, B_2; B_3, B_4)$.

Figure 12 illustrates the previous corollary.

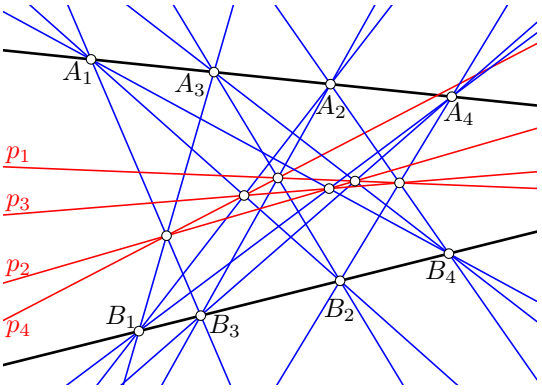

FIGURE 12. The generalized Pappus configuration: The Pappus lines p_i coincide if and only if $\mathrm{CR}(A_1, A_2; A_3, A_4) = \mathrm{CR}(B_1, B_2; B_3, B_4)$.

A weaker form of this statement is Lemma 3.8 in [7].

We get another interesting way of putting Theorem 5.1 (ii) by focussing on the two triangles $A_1A_4(g_1 \times g_4)$ and $A_2A_3(g_2 \times g_3)$:

Corollary 6.3. *Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles. Then the following statements are equivalent:*

(i) *$A_1B_1C_1$ and $A_2B_2C_2$ are perspective from a point.*
(ii) *$A_1B_1C_1$ and $A_2B_2C_2$ are perspective from a line.*
(iii) $\mathrm{CR}(A_1, B_2; A_1', B_2') = \mathrm{CR}(A_2, B_1; A_2', B_1')$ *where* $A_1' = (A_1 \times B_2) \times (B_1 \times C_1)$, $B_1' = (A_2 \times B_1) \times (A_1 \times C_1)$, $A_2' = (A_2 \times B_1) \times (B_2 \times C_2)$, $B_2' = (A_1 \times B_2) \times (A_2 \times C_2)$ *(see Figure 13).*
(iv) $\mathrm{CR}(B_1, C_2; B_1', C_2') = \mathrm{CR}(B_2, C_1; B_2', C_1')$ *where* $B_1' = (B_1 \times C_2) \times (C_1 \times A_1)$, $C_1' = (B_2 \times C_1) \times (B_1 \times A_1)$, $B_2' = (B_2 \times C_1) \times (C_2 \times A_2)$, $C_2' = (B_1 \times C_2) \times (B_2 \times A_2)$.
(v) $\mathrm{CR}(C_1, A_2; C_1', A_2') = \mathrm{CR}(C_2, A_1; C_2', A_1')$ *where* $C_1' = (C_1 \times A_2) \times (A_1 \times B_1)$, $A_1' = (C_2 \times A_1) \times (C_1 \times B_1)$, $C_2' = (C_2 \times A_1) \times (A_2 \times B_2)$, $A_2' = (C_1 \times A_2) \times (C_2 \times B_2)$.

So Corollary 6.3 is Desargues' Theorem with an additional quantitative equivalence that tells us when the perspectives occur.

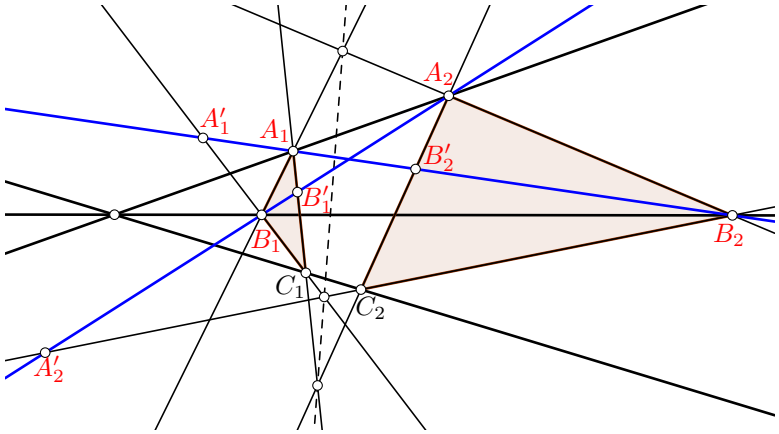

FIGURE 13. Quantitative version of Desargue's Theorem in Corollary 6.3(iii).

Next, we want to demonstrate how we can use Theorem 5.1 (ii) to expand the special case $n = 4$ of the generalized Ceva's Theorem for n-gons which can be found in [4, Theorem 2]. For $n = 4$, the theorem reads as follows:

Theorem 6.1 (Ceva's Theorem for quadrilaterals)**.** *Let A_1, A_2, A_3, A_4 be the vertices of a quadrilateral and P a given point. For $i \in \{1, 2, 3, 4\}$ let $D_i = (A_i \times P) \times (A_{i-1} \times A_{i+1})$. Then,*

$$\frac{A_1D_2}{D_2A_3} \cdot \frac{A_2D_3}{D_3A_4} \cdot \frac{A_3D_4}{D_4A_1} \cdot \frac{A_4D_1}{D_1A_2} = 1. \tag{23}$$

Note that the implication is only one-sided. But now Theorem 5.1 (ii) tells us how we can generalize Theorem 6.1 to get an equivalence:

Theorem 6.2 (Two-sided Ceva's Theorem for quadrilaterals)**.** *Let A_1, A_2, A_3, A_4 be the vertices of a quadrilateral and let g_1, g_2, g_3, g_4 be lines such that for each $i \in \{1, 2, 3, 4\}$, g_i passes through A_i. Then the following statements are equivalent:*

(i) $\dfrac{A_1D_2}{D_2A_3}\cdot\dfrac{A_2D_3}{D_3A_4}\cdot\dfrac{A_3D_4}{D_4A_1}\cdot\dfrac{A_4D_1}{D_1A_2}=1$
where $D_i = g_i \times (A_{i-1} \times A_{i+1})$ *for* $i \in \{1,2,3,4\}$.

(ii) $g_1 \times g_2, g_3 \times g_4, (A_1 \times A_4) \times (A_2 \times A_3)$ *are collinear (see Figure 14).*

(iii) $g_1 \times g_4, g_2 \times g_3, (A_1 \times A_2) \times (A_3 \times A_4)$ *are collinear.*

It is easy to see that when g_1, g_2, g_3, g_4 are concurrent, (ii) and (iii) are trivially satisfied, hence Theorem 6.2 is in fact a generalization of Theorem 6.1.

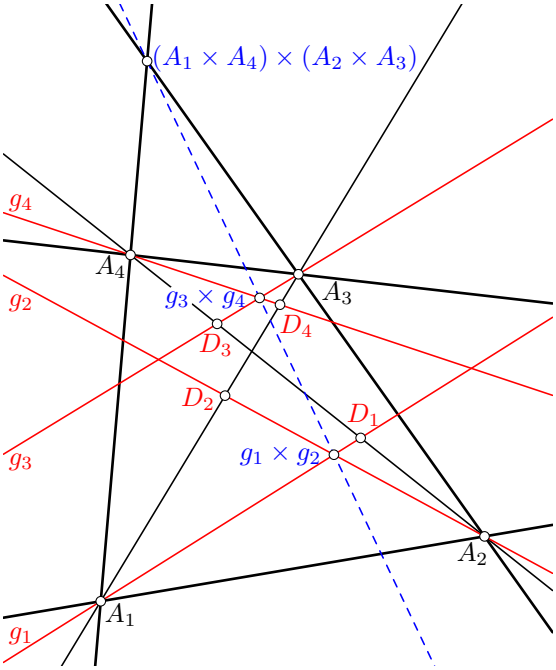

FIGURE 14. Collinear points in Theorem 6.2(ii).

Can the cases $n > 4$ also be generalized in a similar fashion? The answer is yes:

Theorem 6.3 (Two-sided Ceva's Theorem for n-gons)**.** *For* $n \geq 3$*, let* $P^n = [A_1^n, \ldots, A_n^n]$ *be an* n*-gon and let* $g_1^n \ldots, g_n^n$ *be lines such that for each* $i \in \{1, \ldots, n\}$*,* g_i^n *passes through* A_i^n*. For* $m \in \{4, \ldots, n\}$*, the* m*-gon* P^m *can be reduced in a step to the* $(m-1)$*-gon* P^{m-1} *by obeying the following procedure:*

- *Choose two adjacent vertices* A_i^m *and* A_{i+1}^m*,* $i \in \{1, \ldots, m\}$.
- *Replace* A_i^m *and* A_{i+1}^m *by*

$$A_i^{m-1} = (A_{i-1}^m \times A_i^m) \times (A_{i+1}^m \times A_{i+2}^m).$$

- *Replace* g_i^m *and* g_{i+1}^m *by*

$$g_i^{m-1} = A_i^{m-1} \times (g_i^m \times g_{i+1}^m).$$

- *For the other vertices, let*

$$A_j^{m-1} = \begin{cases} A_j^m & \text{for } j \in \{1, \dots, i-1\}, \\ A_{j+1}^m & \text{for } j \in \{i+1, \dots, m-1\}. \end{cases}$$

- *Similarly, for the other lines, let*

$$g_j^{m-1} = \begin{cases} g_j^m & \text{for } j \in \{1, \dots, i-1\}, \\ g_{j+1}^m & \text{for } j \in \{i+1, \dots, m-1\}. \end{cases}$$

If it is possible to reduce P^n in $(n-3)$ steps to a triangle P^3 for which g_1^3, g_2^3, g_3^3 are concurrent, then every sequence of $(n-3)$ steps will lead to such a triangle (see Figure 15). Furthermore, this is the case if and only if

$$\frac{A_1^n D_2^n}{D_2^n A_3^n} \cdot \frac{A_2^n D_3^n}{D_3^n A_4^n} \cdot \ldots \cdot \frac{A_n^n D_1^n}{D_1^n A_2^n} = 1, \tag{24}$$

where $D_i^n = g_i^n \times (A_{i-1}^n \times A_{i+1}^n)$ for $i \in \{1, \dots, n\}$.

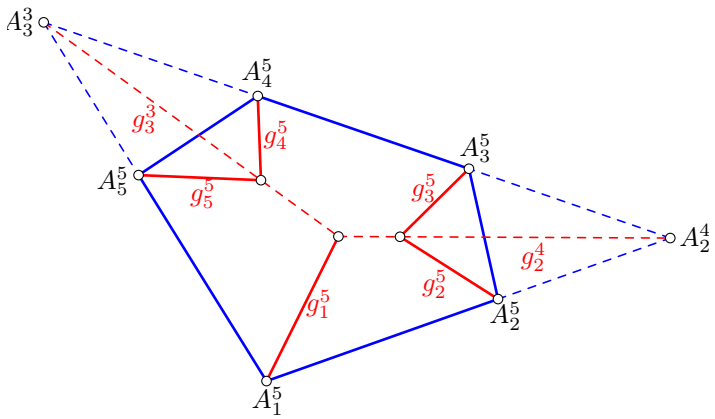

FIGURE 15. Ceva's Theorem for a pentagon $A_1^5 \dots A_5^5$.

In the proof we will make use of the area principle that can also be found in [4]:

Lemma 6.1 (Area principle, see Figure 16)**.** *Whenever D is the intersection point of two lines $A_1 \times A_2$ and $B \times C$, then there holds*

$$\frac{A_1 D}{D A_2} = \frac{[A_1 B C]}{[B A_2 C]}. \tag{25}$$

Proof. Each side of (25) is equal to the ratio of the two heights of the triangles A_1BC and BA_2C with respect to the base BC.

At some point, we will also need a slightly more refined version of the area principle:

Lemma 6.2 (Area principle for unequal bases, see Figure 16)**.** *Whenever D is the intersection point of two lines $A_1 \times A_2$ and $B \times C_1$, on which also the point C_2 lies, then there holds*

$$\frac{A_1 D}{D A_2} = \frac{[A_1 B C_1]}{[B A_2 C_2]} \cdot \frac{B C_2}{B C_1}. \tag{26}$$

Proof. As above, we have

$$\frac{A_1D}{DA_2} = \frac{[A_1BC_2]}{[BA_2C_2]} = \frac{[A_1BC_1]\cdot BC_2/BC_1}{[BA_2C_2]} = \frac{[A_1BC_1]}{[BA_2C_2]}\cdot\frac{BC_2}{BC_1}.$$

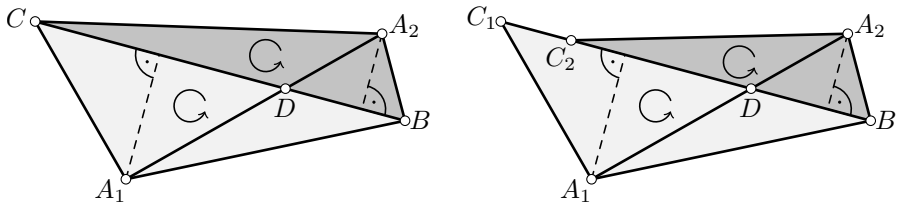

FIGURE 16. Area principles.

Proof.[Proof of Theorem 6.3] We want to show that in every step, the left-hand side of (24) is constant, i.e.:

(27)
$$\frac{A_1^mD_2^m}{D_2^mA_3^m}\cdot\frac{A_2^mD_3^m}{D_3^mA_4^m}\cdot\ldots\cdot\frac{A_m^mD_1^m}{D_1^mA_2^m} = \frac{A_1^{m-1}D_2^{m-1}}{D_2^{m-1}A_3^{m-1}}\cdot\frac{A_2^{m-1}D_3^{m-1}}{D_3^{m-1}A_4^{m-1}}\cdot\ldots\cdot\frac{A_{m-1}^{m-1}D_1^{m-1}}{D_1^{m-1}A_2^{m-1}}$$

By symmetry we can assume that in this step, A_{m-1}^m and A_m^m are replaced by A_{m-1}^{m-1} (this will make the notation easier since $A_j^{m-1} = A_j^m$ for all $j \in \{1,\ldots,m-2\}$). Having a closer look at (27), we see that it is enough to show

$$\begin{aligned}(28)\quad \frac{A_{m-3}^mD_{m-2}^m}{D_{m-2}^mA_{m-1}^m}\cdot\frac{A_{m-2}^mD_{m-1}^m}{D_{m-1}^mA_m^m}\cdot\frac{A_{m-1}^mD_m^m}{D_m^mA_1^m}\cdot\frac{A_m^mD_1^m}{D_1^mA_2^m} = \\ = \frac{A_{m-3}^{m-1}D_{m-2}^{m-1}}{D_{m-2}^{m-1}A_{m-1}^{m-1}}\cdot\frac{A_{m-2}^{m-1}D_{m-1}^{m-1}}{D_{m-1}^{m-1}A_1^{m-1}}\cdot\frac{A_{m-1}^{m-1}D_1^{m-1}}{D_1^{m-1}A_2^{m-1}}\end{aligned}$$

Let G_{m-2} and G_1 be the intersections of g_{m-2}^m and g_1^m with the new line g_{m-1}^{m-1}. Further, let $G_{m-1} = g_{m-1}^m \times g_m^m$. By definition, G_{m-1} also lies on g_{m-1}^{m-1} (see Figure 17).

By using the area principle, we can rewrite all the fractions in (28). Let us start with the left-hand side:

$$(29)\qquad \frac{A_{m-3}^mD_{m-2}^m}{D_{m-2}^mA_{m-1}^m} = \frac{[A_{m-3}^mA_{m-2}^mG_{m-2}]}{[A_{m-2}^mA_{m-1}^mG_{m-2}]}$$

$$(30)\qquad \frac{A_{m-2}^mD_{m-1}^m}{D_{m-1}^mA_m^m} = \frac{[A_{m-2}^mA_{m-1}^mG_{m-1}]}{[A_{m-1}^mA_m^mG_{m-1}]}$$

$$(31)\qquad \frac{A_{m-1}^mD_m^m}{D_m^mA_1^m} = \frac{[A_{m-1}^mA_m^mG_{m-1}]}{[A_m^mA_1^mG_{m-1}]}$$

$$(32)\qquad \frac{A_m^mD_1^m}{D_1^mA_2^m} = \frac{[A_m^mA_1^mG_1]}{[A_1^mA_2^mG_1]}$$

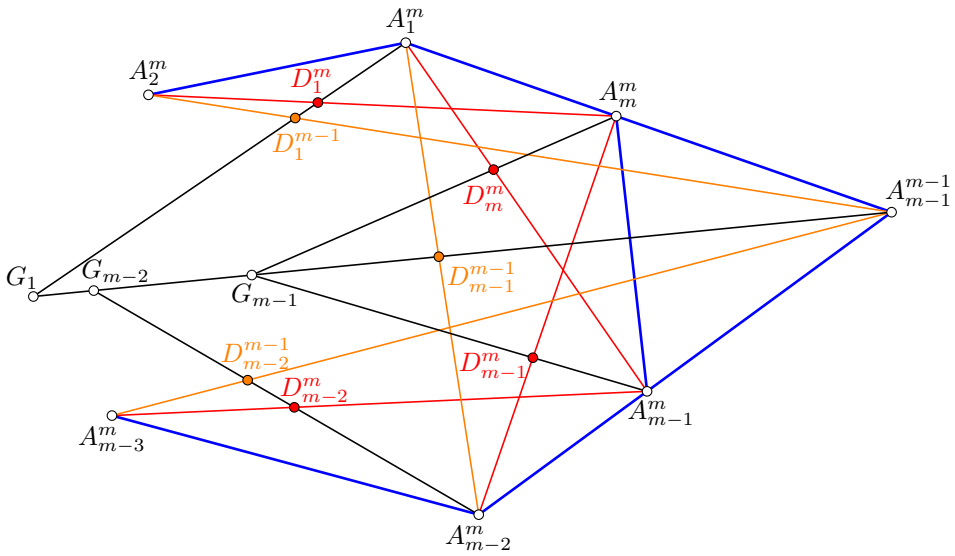

FIGURE 17. Proof of Theorem 6.3.

Also, note that:

$$\frac{[A^m_{m-2}A^m_{m-1}G_{m-1}]}{[A^m_{m-2}A^m_{m-1}G_{m-2}]} = \frac{A^{m-1}_{m-1}G_{m-1}}{A^{m-1}_{m-1}G_{m-2}} \tag{33}$$

$$\frac{[A^m_m A^m_1 G_1]}{[A^m_m A^m_1 G_{m-1}]} = \frac{A^{m-1}_{m-1}G_1}{A^{m-1}_{m-1}G_{m-1}} \tag{34}$$

Now by putting all of this together, we get that the left-hand side of (28) is equal to

$$\frac{[A^m_{m-3}A^m_{m-2}G_{m-2}]}{[A^m_1 A^m_2 G_1]} \cdot \frac{A^{m-1}_{m-1}G_1}{A^{m-1}_{m-1}G_{m-2}}. \tag{35}$$

Let us do the same for the right-hand side. Rewriting the fractions gives us:

$$\frac{A^{m-1}_{m-3}D^{m-1}_{m-2}}{D^{m-1}_{m-2}A^{m-1}_{m-1}} = \frac{[A^{m-1}_{m-3}A^{m-1}_{m-2}G_{m-2}]}{[A^{m-1}_{m-2}A^{m-1}_{m-1}G_{m-2}]} \tag{36}$$

$$\frac{A^{m-1}_{m-2}D^{m-1}_{m-1}}{D^{m-1}_{m-1}A^{m-1}_1} = \frac{[A^{m-1}_{m-2}A^{m-1}_{m-1}G_{m-2}]}{[A^{m-1}_{m-1}A^{m-1}_1 G_1]} \cdot \frac{A^{m-1}_{m-1}G_1}{A^{m-1}_{m-1}G_{m-2}} \tag{37}$$

$$\frac{A^{m-1}_{m-1}D^{m-1}_1}{D^{m-1}_1 A^{m-1}_2} = \frac{[A^{m-1}_{m-1}A^{m-1}_1 G_1]}{[A^{m-1}_1 A^{m-1}_2 G_1]} \tag{38}$$

Hence the right-hand side of (28) is equal to

$$\frac{[A^{m-1}_{m-3}A^{m-1}_{m-2}G_{m-2}]}{[A^{m-1}_1 A^{m-1}_2 G_1]} \cdot \frac{A^{m-1}_{m-1}G_1}{A^{m-1}_{m-1}G_{m-2}}, \tag{39}$$

which is equal to (35). We have proven now that in every step, the left-hand side of (24) is constant.

Now assume that there exists a sequence of $(n-3)$ steps which reduces P^n to a triangle P^3 for which g_1^3, g_2^3, g_3^3 are concurrent. Then by Ceva,

$$\frac{A_1^3 D_2^3}{D_2^3 A_3^3} \cdot \frac{A_2^3 D_3^3}{D_3^3 A_1^3} \cdot \frac{A_3^3 D_1^3}{D_1^3 A_2^3} = 1. \tag{40}$$

But as the product is constant, it must have been 1 right from the start. This implies that also every other sequence of $(n-3)$ steps ends up with a triangle for which this product is still 1, hence by Ceva the lines g_1^3, g_2^3, g_3^3 of every such possible triangle are concurrent. This completes the proof.

We will call lines $g_1^n, \ldots, g_n^n$ of an n-gon with the property as in Theorem 6.3 *pseudo-concurrent*. Note that concurrency always implies pseudo-concurrency. Also, note that the order of the lines $g_1^n, \ldots, g_n^n$ respectively of the vertices $A_1^n, \ldots, A_n^n$ matters, i.e. if the lines $g_1^4, g_2^4, g_3^4, g_4^4$ of the quadrilateral $A_1^4 A_2^4 A_3^4 A_4^4$ are pseudo-concurrent, then this does not imply that the lines $g_1^4, g_3^4, g_2^4, g_4^4$ of the quadrilateral $A_1^4 A_3^4 A_2^4 A_4^4$ are pseudo-concurrent.

A rather trivial but still quite nice consequence of this definition is the following result:

Theorem 6.4. *The internal angle bisectors of every n-gon are pseudo-concurrent.*

Proof. Using the notation of Theorem 6.3, this theorem follows by observing that whenever we replace some g_i^m and g_{i+1}^m by a new line g_i^{m-1}, this new line will thanks to its definition also be an internal angle bisector of the reduced $(m-1)$-gon P^{m-1} (this is also true if P^m is not convex). After $(n-3)$ steps, we get a triangle P^3 and lines g_1^3, g_2^3, g_3^3 which are the internal angle bisectors of this triangle and hence are concurrent. Thus, the internal angle bisectors of the original n-gon are pseudo-concurrent.

It is possible to consider the external angle bisectors as well and get the following version:

Theorem 6.5. *Let $A_1, \ldots, A_n$ be an n-gon. For every $i \in \{1, \ldots, n\}$, choose one of the two angle bisectors in A_i. If the number of chosen external angle bisectors is even, then the chosen angle bisectors are pseudo-concurrent.*

Proof. This follows by thoroughly checking all cases when replacing g_i^m and g_{i+1}^m by g_i^{m-1} and seeing that

- whenever g_i^m and g_{i+1}^m are both internal or both external angle bisectors, then g_i^{m-1} is an internal angle bisector, and
- whenever one of g_i^m and g_{i+1}^m is an internal and the other an external angle bisector, then g_i^{m-1} is an external angle bisector,

hence the number of external angle bisectors will always stay even. In the end, we get a triangle P^3 and either three internal angle bisectors or one internal and two external angle bisectors, hence the remaining angle bisectors will be concurrent. From this, we get that the original angle bisectors are pseudo-concurrent.

We can also generalize Menelaos in a similar way:

Theorem 6.6 (Two-sided Menelaos' Theorem for n-gons)**.** *For $n \geq 3$, let $P^n = [A_1^n, \ldots, A_n^n]$ be an n-gon and let $B_1^n \ldots, B_n^n$ be points such that for*

each $i \in \{1,\dots,n\}$, B_i^n lies on $A_i^n \times A_{i+1}^n$. For $m \in \{4,\dots,n\}$, the m-gon P^m can be reduced in a step to the $(m-1)$-gon P^{m-1} by obeying the following procedure:

- *Choose a vertex A_i^m, $i \in \{1,\dots,m\}$.*
- *Replace B_{i-1}^m and B_i^m by*

$$B_{i-1}^{m-1} = (A_{i-1}^m \times A_{i+1}^m) \times (B_{i-1}^m \times B_i^m).$$

- *Let*

$$A_j^{m-1} = \begin{cases} A_j^m & \text{for } j \in \{1,\dots,i-1\}, \\ A_{j+1}^m & \text{for } j \in \{i,\dots,m-1\}. \end{cases}$$

- *Let*

$$B_j^{m-1} = \begin{cases} B_j^m & \text{for } j \in \{1,\dots,i-2\}, \\ B_{j+1}^m & \text{for } j \in \{i,\dots,m-1\}. \end{cases}$$

If it is possible to reduce P^n in $(n-3)$ steps to a triangle P^3 for which B_1^3, B_2^3, B_3^3 are collinear, then every sequence of $(n-3)$ steps will lead to such a triangle (see Figure 18). Furthermore, this is the case if and only if

$$\frac{A_1^n B_1^n}{B_1^n A_2^n} \cdot \frac{A_2^n B_2^n}{B_2^n A_3^n} \cdot \ldots \cdot \frac{A_n^n B_n^n}{B_n^n A_1^n} = (-1)^n. \tag{41}$$

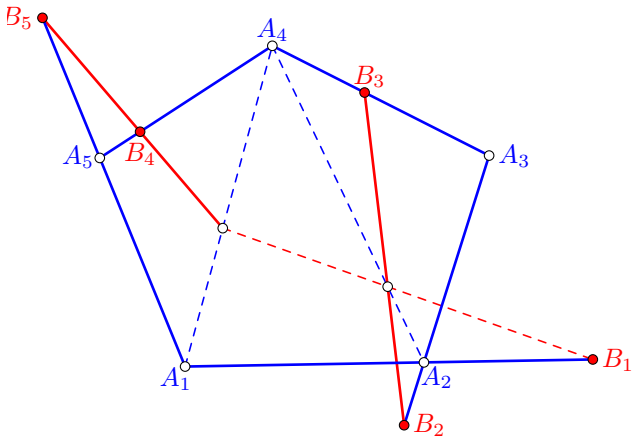

FIGURE 18. Theorem of Menelaos for the pentagon $A_1^5 \dots A_5^5$.

Proof. We will show that in each step, the left-hand side of (41) will only change by a factor -1. Precisely, we have to show that

$$\frac{A_{m-1}^m B_{m-1}^m}{B_{m-1}^m A_m^m} \cdot \frac{A_m^m B_m^m}{B_m^m A_1^m} = -\frac{A_{m-1}^{m-1} B_{m-1}^{m-1}}{B_{m-1}^{m-1} A_1^{m-1}} \tag{42}$$

if the vertex A_m^m was chosen when reducing P^m to P^{m-1}. But this follows directly by applying Menelaos' Theorem to the triangle $A_{m-1}^m A_m^m A_1^m$ (see Figure 19).

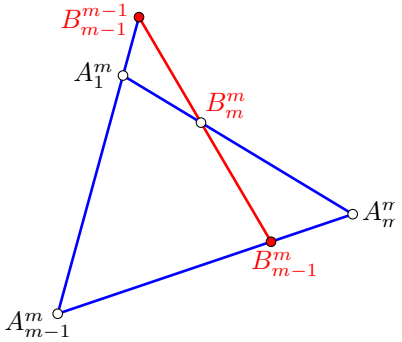

FIGURE 19. Proof of Theorem 6.6.

Hence, if there exists a sequence of $(n-3)$ steps that reduces P^n to a triangle P^3 for which B_1^3, B_2^3, B_3^3 are collinear, then by Menelaos' Theorem,

$$\frac{A_1^3B_1^3}{B_1^3A_2^3} \cdot \frac{A_2^3B_2^3}{B_2^3A_3^3} \cdot \frac{A_3^3B_3^3}{B_3^3A_1^3} = -1. \tag{43}$$

But as the product only changed by a factor -1 in each of the $(n-3)$ steps, we get that

$$\frac{A_1^nB_1^n}{B_1^nA_2^n} \cdot \frac{A_2^nB_2^n}{B_2^nA_3^n} \cdot \ldots \cdot \frac{A_n^nB_n^n}{B_n^nA_1^n} = (-1) \cdot (-1)^{n-3} = (-1)^n. \tag{44}$$

This implies that also every other sequence of $(n-3)$ steps leads to a triangle for which B_1^3, B_2^3, B_3^3 are collinear. This completes the proof.

As before, we suggest to call points $B_1^n, \ldots, B_n^n$ of an n-gon with the property as in Theorem 6.6 *pseudo-collinear*. Again, collinearity implies pseudo-collinearity, and the order of the points matters.

An immediate consequence of these new definitions is the following result:

Corollary 6.4. *If the points $B_1, \ldots, B_n$ on the sides of an n-gon are pseudo-collinear, then the lines $b_1, \ldots, b_n$ in the dual configuration are pseudo-concurrent and vice-versa.*

The new concepts of *pseudo-concurrency* and *pseudo-collinearity* are a very natural way to generalize the well-known terms *concurrency* and *collinearity*. As Theorem 6.4 and Corollary 6.4 show, it's easy to transfer some basic results to these new concepts. We are convinced that many more results can be carried over to this new setting, and are looking forward to whatever results this new perception may inspire.

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