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Brocard families

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Abstract We consider families of triangles which share the same Brocard angle. Historically, such families first occurred in the context of projections of equilateral triangles. We introduce two new Brocard families. The first one is related to triangles that are inscribed in a certain way in a triangle. The second new Borcard family occurs in a configuration related to Routh's theorem.

Keywords Brocard points · Brocard angles · Brocard families

Mathematics Subject Classification 51M04 · 51M15

1 Introduction

The Brocard points are a classic of triangular geometry. They are named after the French mathematician Henri Brocard who investigated these points around 1875 (see [7] for a historical note, and [6, p. 10] or [1, p. 48–52] for a modern presentation). Recall that the first Brocard point of a triangle ABC is a point P such that $\sphericalangle BAP = \sphericalangle CBP = \sphericalangle ACP$ (see Fig. 1). To see that such a point exists, consider the three circles c_A,c_B,c_C, were c_A passes through A and is tangent to the side BC in B, and similarly for c_B and c_C. Then it follows from Miquel's Theorem (see the original work [8, 9] or [3, Theorem 3.32] for a modern presentation) that the three circles

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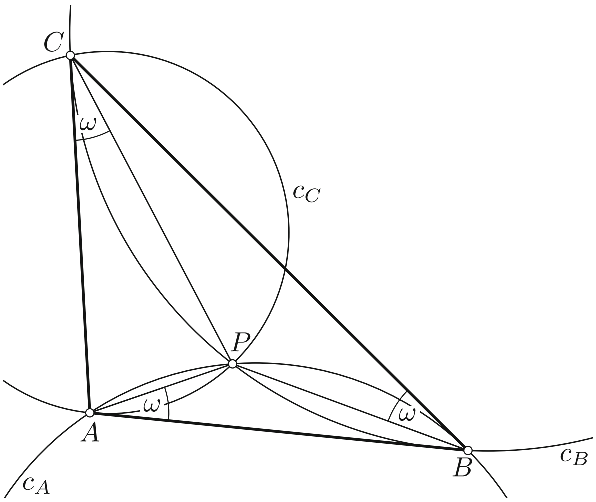


Fig. 1 The first Brocard point P of the triangle ABC

meet in a common point P. By Eucl. III.32[1] it follows that the three angles marked with ω in the figure are equal. The second Brocard point Q is defined similarly by the property that $\sphericalangle ABQ = \sphericalangle BCQ = \sphericalangle CAQ$. It is astonishing that these angles are also equal to ω, the so called Brocard angle of the triangle.

Before we begin, a word about the notation used. We will denote a line through the points A and B by AB, and at the same time we will write AB for the segment AB and its length. It will always be clear from the context what is meant.

2 Brocard families

William Gallatly made in [4, p. 91] an interesting observation. If equilateral triangles are projected orthogonally onto a plane, then all projected triangles have the same Brocard angle (see Fig. 2). The reader is invited to show that the following version is also true: If equilateral triangles are projected by rays orthogonal to the triangle plane onto another plane, then all projected triangles share the same Brocard angle.

In this article we are interested in other families of triangles all sharing the same Brocard angle. Formally we define such a family as follows.

Definition 1 Let $(\triangle_\mu)_\mu$ be a family of triangles in the Euclidean plane, where the parameter μ belongs to a finite or infinite set. Let the sides of the triangle $\triangle_\mu$ be a_μ, b_μ, and c_μ. Then, $(\triangle_\mu)_\mu$ is called a Brocard family, if all triangles $\triangle_\mu$ share the same Brocard angle.

Brocard families can easily be characterised by the following arithmetic criterion.

[1] The chord-tangent angle is the same size as an opposite inscribed angle.

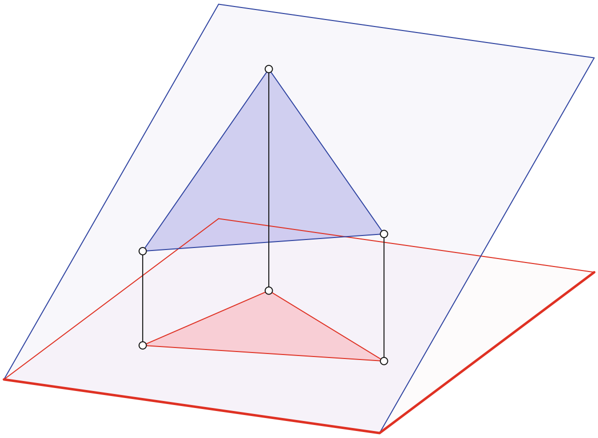


Fig. 2 The blue equilateral triangle is orthogonally projected to the red triangle

Lemma 1 $(\triangle_\mu)_\mu$ is a Brocard family if and only if the value

$$\frac{a_\mu^4 + b_\mu^4 + c_\mu^4}{(a_\mu^2 + b_\mu^2 + c_\mu^2)^2}$$

is a constant independent of μ.

Proof The Brocard angle ω of the triangle $\triangle_\mu$ is given by

$$\tan(\omega) = \frac{4\,\text{area}(\triangle_\mu)}{a_\mu^2 + b_\mu^2 + c_\mu^2}$$

(see, e.g., [5, p. 90–93] or [6, p. 10]). Using the well known formula

$$\text{area}(\triangle_\mu) = \frac{1}{4}\sqrt{(a_\mu^2 + b_\mu^2 + c_\mu^2)^2 - 2(a_\mu^4 + b_\mu^4 + c_\mu^4)}$$

for the area of the triangle $\triangle_\mu$, we get

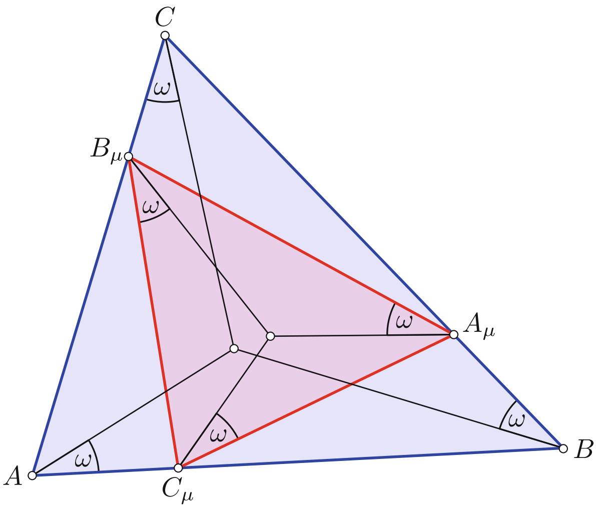
$$\tan(\omega) = \sqrt{1 - 2\frac{a_\mu^4 + b_\mu^4 + c_\mu^4}{(a_\mu^2 + b_\mu^2 + c_\mu^2)^2}}\,.$$

This finishes the proof. □

The first new Brocard family is obtaind by inscribing a triangle $A_\mu B_\mu C_\mu$ in a triangle ABC such that the three vertices A_μ, B_μ, C_μ divide the sides of ABC all in the same ration μ (see Fig. 3). More precisely we have the following theorem.

Theorem 1 Let $A_\mu B_\mu C_\mu$ be a triangle inscribed in the triangle ABC such that A_μ lies on the line BC, B_μ lies on the line CA, C_μ lies on the line AB, and $AC_\mu = \mu\, AB, BA_\mu = \mu\, BC, CB_\mu = \mu\, CA$ for $\mu \in \mathbb{R}$. Then, $(A_\mu B_\mu C_\mu)_\mu$ is a Brocard family which contains the triangle ABC for $\mu = 0$.

Fig. 3 The points $\boldsymbol{A}_\mu, \boldsymbol{B}_\mu, \boldsymbol{C}_\mu$ divide the sides of the triangle in the same ratio. Then the Brocard angle ω is the same for the triangles ABC and $\boldsymbol{A}_\mu \boldsymbol{B}_\mu \boldsymbol{C}_\mu$



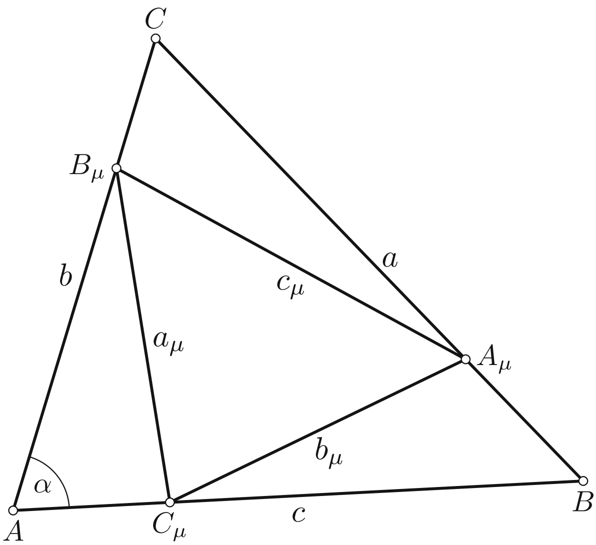
Proof By the law of cosines, applied to the triangles ABC and $AC_\mu B_\mu$ respectively, we have

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$a_\mu^2 = (1-\mu)^2 b^2 + \mu^2 c^2 - 2\mu(1-\mu)bc\ \cos(\alpha)$$

where we adopt the notation from Fig. 4. Eliminating $\cos(\alpha)$ from the two equations yields

$$a_\mu^2 = \mu(1-\mu)a^2 + (1-3\mu+2\mu^2)b^2 + \mu(2\mu-1)c^2.$$

Fig. 4 Proof of Theorem 1



Cyclically, we have also

$$b_\mu^2 = \mu(1-\mu)b^2 + (1-3\mu+2\mu^2)c^2 + \mu(2\mu-1)a^2$$
$$c_\mu^2 = \mu(1-\mu)c^2 + (1-3\mu+2\mu^2)a^2 + \mu(2\mu-1)b^2.$$

If we add up the four squares of the sides, we get

$$a_\mu^2 + b_\mu^2 + c_\mu^2 = (a^2+b^2+c^2)(1-3\mu+3\mu^2).$$

Squaring the squares we obtain for the sum of the fourth powers

$$a_\mu^4 + b_\mu^4 + c_\mu^4 = (a^4+b^4+c^4)(1-3\mu+3\mu^2)^2$$

and it follows that

$$\frac{a_\mu^4 + b_\mu^4 + c_\mu^4}{(a_\mu^2 + b_\mu^2 + c_\mu^2)^2} = \frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}.$$

Now the claim follows immediately from the lemma. □

The second new Brocard family we want to present is related to the first one in that the points A_μ, B_μ, and C_μ on the sides of the triangle ABC again play a decisive role. This time, we consider the triangle with sides AA_μ, BB_μ, CC_μ (see Fig. 5). This configuration occurs in Routh's theorem (see [2, Theorem 13.55]) if the same division ratio applies on all three sides.

It turns out that these triangles also form a Brocard family, and we have the following theorem.

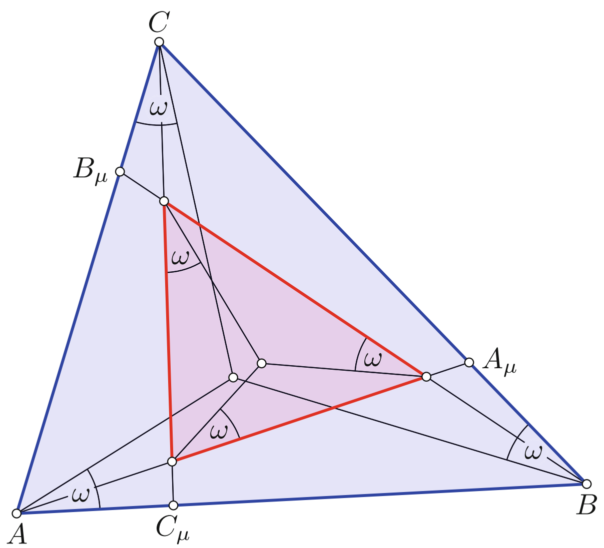


Fig. 5 Theorem 2: The red triangles form a Brocard family

Theorem 2 Let A_μ, B_μ, C_μ be the points on the side of a triangle ABC as specified in Theorem 1. Then the triangles with the lines AA_μ, BB_μ, CC_μ as sides form a Brocard family which contains the triangle ABC for $\mu = 0$.

Proof By the law of cosines, applied to the triangles ABC and $AC_\mu C$ respectively, we have

$$a^2 = b^2 + c^2 - 2bc\cos(\alpha)$$
$$CC_\mu^2 = b^2 + \mu^2 c^2 - 2\mu bc\cos(\alpha)$$

where we adopt the notation from Fig. 6. Eliminating $\cos(\alpha)$ from the two equations yields for the square of the length of the Cevian CC_μ

$$CC_\mu^2 = (1-\mu)b^2 + \mu(\mu-1)c^2 + \mu a^2.$$

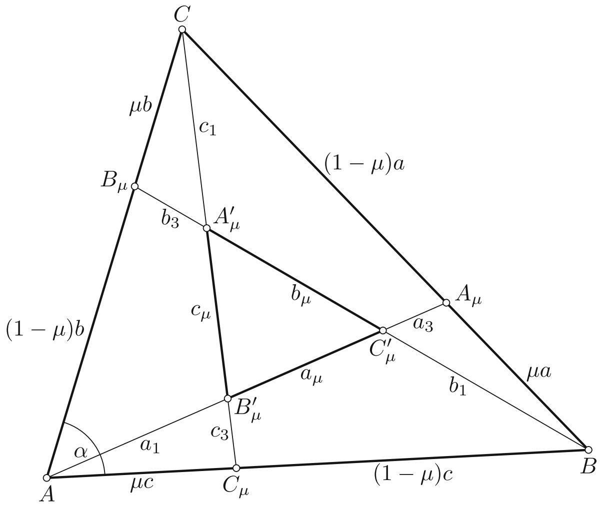
Cyclically, we obtain

$$AA_\mu^2 = (1-\mu)c^2 + \mu(\mu-1)a^2 + \mu b^2 \tag{1}$$
$$BB_\mu^2 = (1-\mu)a^2 + \mu(\mu-1)b^2 + \mu c^2. \tag{2}$$

We denote the vertices of the triangle with the lines AA_μ, BB_μ, and CC_μ as sides by A'_μ, B'_μ, C'_μ, as indicated in Fig. 6. In order to apply the criterion formulated in the lemma, we need to calculate the length of the sides a_μ, b_μ, and c_μ of this triangle. To do so, we now use the theorem of Menelaus in the triangle ABA_μ with the line CC_μ. We obtain

$$\frac{AC_\mu}{C_\mu B} \cdot \frac{BC}{CA_\mu} \cdot \frac{A_\mu B'_\mu}{B'_\mu A} = -1.$$

Fig. 6 Proof of Theorem 2



This can be written as

$$\frac{A_\mu B'_\mu}{B'_\mu A} = \frac{C_\mu B}{AC_\mu} \cdot \frac{CA_\mu}{CB} = \frac{(1-\mu)^2}{\mu}.$$

Using the notation from Fig. 6, this can be reformulated as

$$\frac{a_\mu + a_3}{a_1} = \frac{(1-\mu)^2}{\mu}$$

and hence

$$\frac{AA_\mu}{a_1} = \frac{a_1 + a_\mu + a_3}{a_1} = 1 + \frac{a_\mu + a_3}{a_1} = 1 + \frac{(1-\mu)^2}{\mu} = \mu - 1 + \frac{1}{\mu}. \tag{3}$$

Accordingly, the ratios $\frac{BB_\mu}{b_1}$ and $\frac{CC_\mu}{c_1}$ have the same value. Again by the theorem of Menelaus, this time applied to the triangle ABC'_μ and the line CC_μ, we get:

$$\frac{AC_\mu}{C_\mu B} \cdot \frac{BA'_\mu}{A'_\mu C'_\mu} \cdot \frac{C'_\mu B'_\mu}{B'_\mu A} = -1$$

and hence

$$\frac{BA'_\mu}{A'_\mu C'_\mu} \cdot \frac{C'_\mu B'_\mu}{AB'_\mu} = \frac{C_\mu B}{AC_\mu} = \frac{1-\mu}{\mu}.$$

This can be written as

$$\frac{b_1 + b_\mu}{b_\mu} \cdot \frac{a_\mu}{a_1} = \frac{1-\mu}{\mu}. \tag{4}$$

Cyclically, we have

$$\frac{c_1 + c_\mu}{c_\mu} \cdot \frac{b_\mu}{b_1} = \frac{1-\mu}{\mu} \tag{5}$$

$$\frac{a_1 + a_\mu}{a_\mu} \cdot \frac{c_\mu}{c_1} = \frac{1-\mu}{\mu}. \tag{6}$$

Solving the Eqs. (4), (5) and (6) for a_μ, b_μ, c_μ gives

$$a_\mu = a_1\Big(\frac{1}{\mu} - 2\Big), \quad b_\mu = b_1\Big(\frac{1}{\mu} - 2\Big), \quad c_\mu = c_1\Big(\frac{1}{\mu} - 2\Big). \tag{7}$$

Now, from Eqs. (3) and (7) we get

$$a_\mu = a_1\big(\frac{1}{\mu} - 2\big) = \frac{AA_\mu}{\mu - 1 + \frac{1}{\mu}}\big(\frac{1}{\mu} - 2\big) = AA_\mu \frac{1-2\mu}{1-\mu+\mu^2}. \tag{8}$$

Taking the square in Eq. (8) and using Eq. (1) we have

$$a_\mu^2 = \big(\frac{1-2\mu}{1-\mu+\mu^2}\big)^2\big((1-\mu)c^2 + \mu(\mu-1)a^2 + \mu b^2\big)$$

and cyclically

$$b_\mu^2 = \big(\frac{1-2\mu}{1-\mu+\mu^2}\big)^2\big((1-\mu)a^2 + \mu(\mu-1)b^2 + \mu c^2\big)$$
$$c_\mu^2 = \big(\frac{1-2\mu}{1-\mu+\mu^2}\big)^2\big((1-\mu)b^2 + \mu(\mu-1)c^2 + \mu a^2\big).$$

Summing up the last three equations, we obtain

$$a_\mu^2 + b_\mu^2 + c_\mu^2 = (a^2 + b^2 + c^2)\frac{(1-2\mu)^2}{1-\mu+\mu^2}.$$

Squaring the squares we find for the sum of the fourth powers

$$a_\mu^4 + b_\mu^4 + c_\mu^4 = (a^4 + b^4 + c^4)\frac{(1-2\mu)^4}{(1-\mu+\mu^2)^2}$$

and it follows that

$$\frac{a_\mu^4 + b_\mu^4 + c_\mu^4}{(a_\mu^2 + b_\mu^2 + c_\mu^2)^2} = \frac{a^4 + b^4 + c^4}{(a^2 + b^2 + c^2)^2}.$$

Now the claim follows immediately from the lemma and we are done. □

By the calculations in the previous proof, we have

$$AA_\mu : a_1 : a_\mu : a_3 = (1 - \mu + \mu^2) : \mu : (1-2\mu) : \mu^2$$

and the same ratios apply for the segments on BB_μ and CC_μ. Hence we obtain as a corollary the following.

Corollary 1 The triangles made from the sides AA_μ, BB_μ, CC_μ form a Brocard family. The triangles made from the sides a_1, b_1, c_1 form a Brocard family. The triangles made from the sides a_3, b_3, c_3 form a Brocard family.

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Conflict of interest N. Hungerbühler and Y. Zhang declare that they have no competing interests.

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