Discretization of Generalized Convection-Diffusion

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Seminar für Angewandte Mathematik
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Generalized Convection-Diffusion

scalar convection diffusion:

$$-\varepsilon\Delta u + \boldsymbol{\beta} \cdot \mathbf{grad} u = f \quad \text{in } \Omega$$

in Differential Forms:

$$d{*}d\omega_0 + {*}L_{\boldsymbol{\beta}}\omega_0 = f \quad \text{in } \Omega$$

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scalar convection diffusion:

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in Differential Forms:

0 form

$$d{*}d\omega_0 + {*}L_{\boldsymbol{\beta}}\omega_0 = f \quad \text{in } \Omega$$

exterior derivative

Hodge operator

Lie derivative

Goal: convection diffusion for p forms ω_p

What is a Lie derivative?

$$L_{\boldsymbol{\beta}} = ?$$

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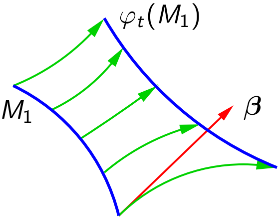
Lie derivatives

directional derivative

$$(\boldsymbol{\beta} \cdot \mathbf{grad} u)(\mathbf{x}) = \lim_{t \to \infty} \frac{u(\mathbf{x} + t\boldsymbol{\beta}) - u(\mathbf{x})}{t}$$



Lie derivative $L_{\boldsymbol{\beta}}$ (transport of forms)
with respect to flow φ_t of velocity field $\boldsymbol{\beta}$:

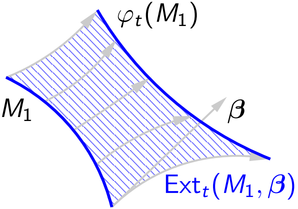
$$\int_{M_p} L_{\boldsymbol{\beta}} \omega_p := < L_{\boldsymbol{\beta}} \omega_p, M_p > := \lim_{t \to 0} \frac{< \omega_p, \varphi_t(M_p) > - < \omega_p, M_p >}{t}$$

Cartan magic formula

$$L_{\boldsymbol{\beta}} = i_{\boldsymbol{\beta}} d + d i_{\boldsymbol{\beta}}$$

with contraction $i_{\boldsymbol{\beta}}$ (Bossavit)

$$< i_{\boldsymbol{\beta}} \omega_p, M_{p-1} > := \lim_{t \to 0} \frac{< \omega_p, Ext_t(M_{p-1}, \boldsymbol{\beta}) >}{t}$$



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generalized convection diffusion for p form ω_p

$$d * d\omega_p + *L_{\boldsymbol{\beta}}\omega_p = f_p$$

or

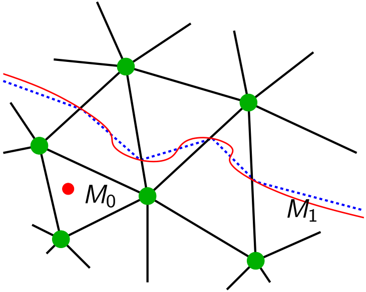
$$d * d\omega_p + *i_{\boldsymbol{\beta}} d\omega_p = f_p \quad \text{or} \quad d * d\omega_p + *di_{\boldsymbol{\beta}}\omega_p = f_p$$

example: magnetic convection

$$\mathbf{curlcurlA} + \beta \times \mathbf{curlA} = \mathbf{F}$$

Discrete Differential Forms

differential forms ω_p act on p dimensional manifolds M_p!
$< \omega_p, M_p > := \int_{M_p} \omega_p$



discrete setting:
prescribe ω_p on finitely many $M_p^{\mathbf{k}}$
(vertices $\mathbf{k} = i$, edges $\mathbf{k} = (e_1, e_2)$ ...).

interpolation of $M_p \to$ approximation
$< \omega_p, M_p > \cong \sum_{\mathbf{k}} a_{\mathbf{k}}(M_p) < \omega_p, M_p^{\mathbf{k}} >$

limit procedure $\to$ Whitney forms $\omega_p^{\mathbf{k}}$
$\omega_p(x) \cong \omega_p^h(x) = \sum_{\mathbf{k}} \omega_p^{\mathbf{k}}(x) < \omega_p, M_p^{\mathbf{k}} >, \quad \omega_p^{\mathbf{k}}(x) := \lim_{M_p \to x} a_{\mathbf{k}}(M_p)$

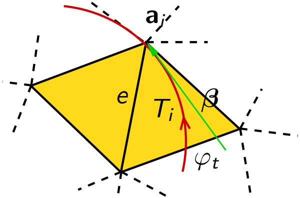
- $p = 0$: ω_0^i Linear Finite Elements
- $p = 1$: $\omega_1^{\mathbf{e}}$ Edge Elements

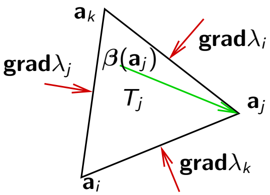
$\Longrightarrow$ back in FEM-setting, but conforming!

Lie derivative of discrete 0-forms

Discrete version of $\boldsymbol{\beta} \cdot \mathbf{grad} \equiv i_{\boldsymbol{\beta}} d \simeq \mathbf{C}\,\mathbf{G} =: \mathbf{L}$?

$$G_{\mathbf{e}i} \overset{\omega_0^i = \lambda_i}{:=} < \mathbf{grad}\lambda_i, \mathbf{e} > \overset{\text{Stokes}}{=} \delta_{ie_2} - \delta_{ie_1}$$





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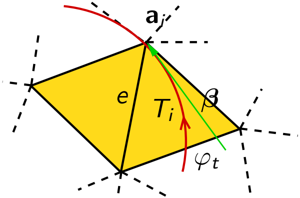
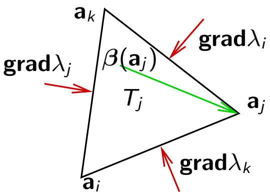
$$
\begin{aligned}
G_{\mathbf{e}i} &\overset{\omega_0^i = \lambda_i}{:=} < \mathbf{grad}\lambda_i, \mathbf{e} > \\
&\overset{\text{Stokes}}{=} \delta_{ie_2} - \delta_{ie_1} \\
C_{i\mathbf{e}} &:= < i_{\boldsymbol{\beta}} \omega_1^{\mathbf{e}}, \mathbf{a}_i > \\
&= \lim_{t \to 0^-} \frac{< \omega_1^{\mathbf{e}}, \mathrm{Ext}_t(\mathbf{a}_i, \boldsymbol{\beta}) >}{t} \\
&\overset{\text{upwind}}{=} \boldsymbol{\beta}(\mathbf{a}_i) \cdot \omega_1^{\mathbf{e}}(\mathbf{a}_i)\big|_{T_i} \\
&= -G_{\mathbf{e}i} \boldsymbol{\beta}(\mathbf{a}_i) \cdot \mathbf{grad}\lambda_{\mathbf{e}/i}\big|_{T_i}
\end{aligned}
$$





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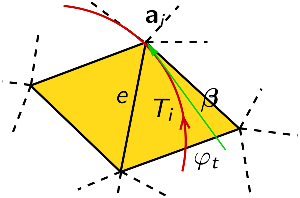

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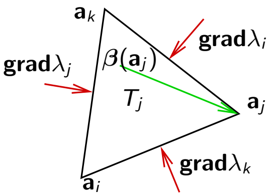
&= -G_{\mathbf{e}i}\boldsymbol{\beta}(\mathbf{a}_i) \cdot \mathbf{grad}\lambda_{\mathbf{e}/i}|_{T_i}
\end{aligned}$$

$$\begin{aligned}
L_{ji} &:= \sum_{\mathbf{e}} C_{j\mathbf{e}} G_{\mathbf{e}i} \\
&= \sum_{\mathbf{e}} -\boldsymbol{\beta}(\mathbf{a}_j) \cdot \mathbf{grad}\lambda_{\mathbf{e}/j}|_{T_j} G_{\mathbf{e}j} G_{\mathbf{e}i} \\
&= \boldsymbol{\beta}(\mathbf{a}_j) \cdot \mathbf{grad}\lambda_i|_{T_j}
\end{aligned}$$

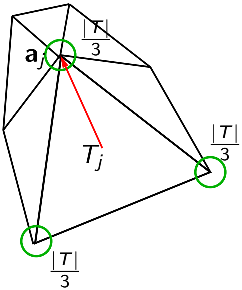


L is M-matrix, inverse monoton!

FEM-Approach

bilinear form and upwind quadrature (Tabata)

$$b(u_h,\lambda_j) := (\boldsymbol{\beta}\cdot\mathbf{grad}\,u_h,\lambda_j)_{L^2} \quad u_h\in P_h^1$$
$$= \sum_T \int_T \boldsymbol{\beta}\cdot\mathbf{grad}\,u_h\lambda_j$$
$$\simeq \boldsymbol{\beta}(\mathbf{a}_j)\cdot\mathbf{grad}\,u_h|_{T_j} \underbrace{\sum_{T\in supp(\lambda_j)}\frac{|T|}{3}}_{\text{discr. Hodge } P_j}$$
$$b_h(\lambda_i,\lambda_j) = P_j\underbrace{\boldsymbol{\beta}(\mathbf{a}_j)\cdot\mathbf{grad}\,\lambda_i|_{T_j}}_{L_{ji}}$$



- error analysis using Strang-Lemma und Bramble-Hilbert techniques.

$$|b_h(u_h,v_h)-b(u_h,v_h)|\leq C\,h|\boldsymbol{\beta}|_{1,\infty}|u_h|_1\|w_h\|_0$$

- discrete Max. principle and L^∞-stability since M-matrix

Numerical Experiments

singular perturbed convection diffusion

$$-\varepsilon \Delta u + \boldsymbol{\beta} \cdot \mathbf{grad} u = f \quad 0 < \varepsilon \ll 1$$

or

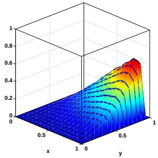
$$d *_{\varepsilon} d\omega_0 + *i_{\boldsymbol{\beta}} d\omega_0 = f \quad 0 < \varepsilon \ll 1$$

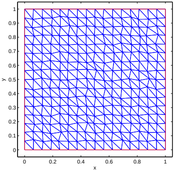
- instability in standard FEM
- upwind finite differences
- artificial viscosity
- Streamline Upwind Petrov Galerkin (SUPG/SDFEM)

Numerical Experiments: Convergence and Stability

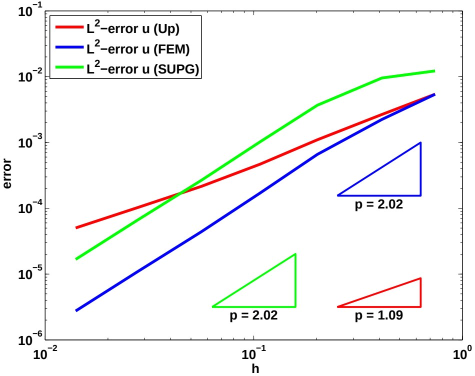
- $\beta_1 = 2$, $\beta_2 = 3$
- force data s.t.

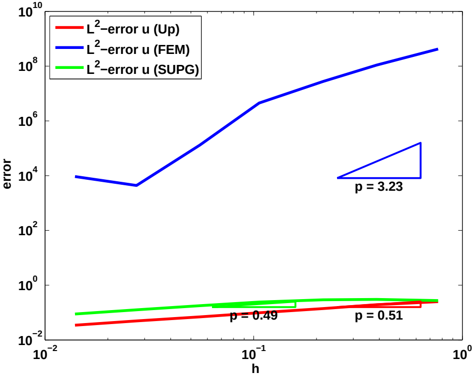
$$u_\varepsilon(x,y) = xy^2 - y^2 e^{2\frac{x-1}{\varepsilon}} - xe^{3\frac{y-1}{\varepsilon}} + e^{2\frac{x-1}{\varepsilon} + 3\frac{y-1}{\varepsilon}}$$





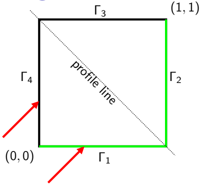
convergence rate with $\varepsilon = 1$ (left) and $\varepsilon = 10^{-10}$ (right)



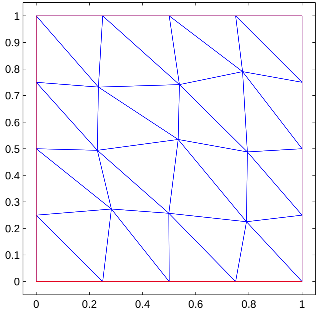


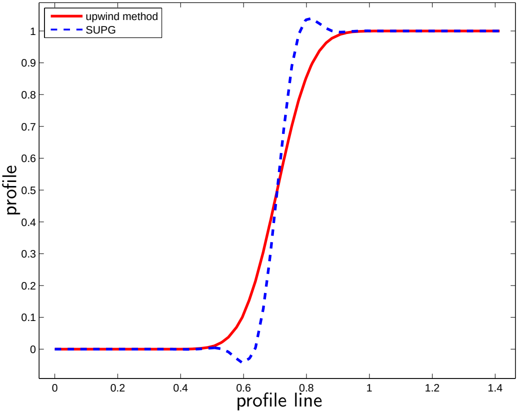
Numerical Experiments: Smoothing

- $\beta_1 = \beta_2 = 1,\ f \equiv 0$
- $u \equiv 1$ on $\Gamma_1 \cup \Gamma_2$
- $u \equiv 0$ on $\Gamma_3 \cup \Gamma_4$

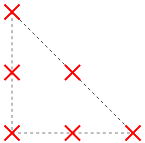


Solution for $\varepsilon = 10^{-14}$ with upwind scheme and SUPG (mesh width=0.027).





Second Order Elements

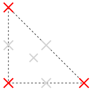


second order Lagrangian elements

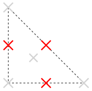
- 6 local basis functions with dofs Σ

4 quadrature rules: $Q(T) = (\mathbf{a}_i, |T|\omega_i)_i$

1. $O(h^2)$



2. $O(h^3)$



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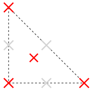
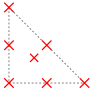


4. $O(h^4)$



bilinearform

$$b_h(u_h, v_h) = \sum_T |T| \sum_{\mathbf{a}_i \in Q(T)} \omega_i (\boldsymbol{\beta} \cdot \mathbf{grad}\, u_h)|_{T_i}(\mathbf{a}_i) v_h(\mathbf{a}_i)$$

$$= \underbrace{\sum_{\mathbf{a}_i \in \Sigma} v_i \omega_i (\boldsymbol{\beta} \cdot \mathbf{grad}\, u_h)|_{T_i}(\mathbf{a}_i) \sum_{T : \mathbf{a}_i \in T} |T|}_{:=\text{element boundary contribution}} + \underbrace{\sum_T |T| \sum_{\mathbf{a}_i \notin \Sigma} \omega_i (\boldsymbol{\beta} \cdot \mathbf{grad}\, u_h)|_T(\mathbf{a}_i) v_h(\mathbf{a}_i)}_{:=\text{element center contribution}}$$

Numerical Experiments: Convergence and Stability

- $\beta_1 = 2,\ \beta_2 = 3$
- force data s.t.

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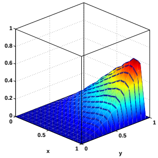

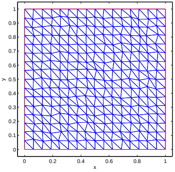

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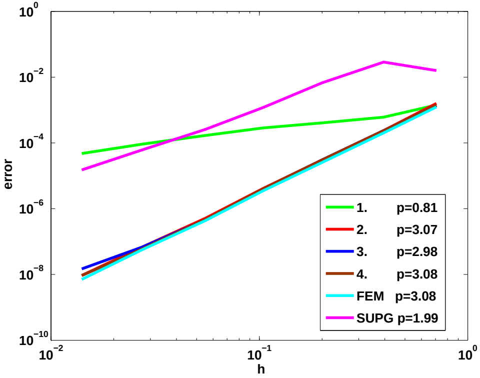
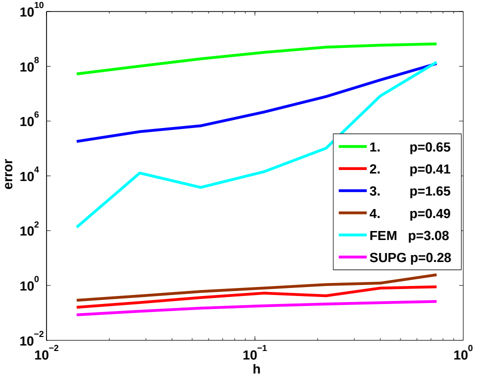
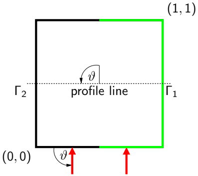
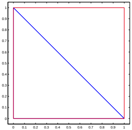




Numerical Experiments: Smoothing

- $\beta_1 = cos(\vartheta)$, $\beta_2 = sin(\vartheta)$, $f \equiv 0$
- $u \equiv 1$ on Γ_1, $u \equiv 0$ on Γ_2
- $\varepsilon = 10^{-14}$, $h = 0.042$





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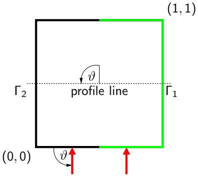

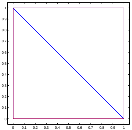

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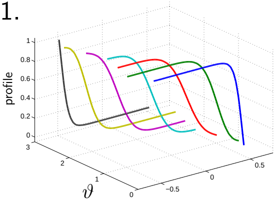

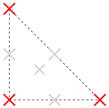

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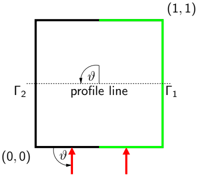

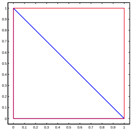

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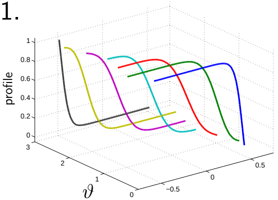

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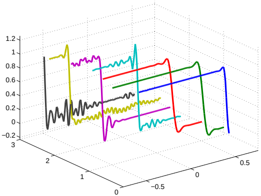

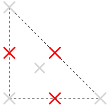

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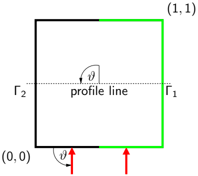

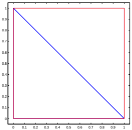

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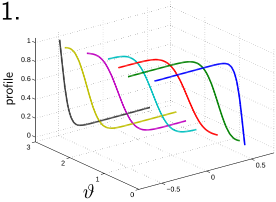

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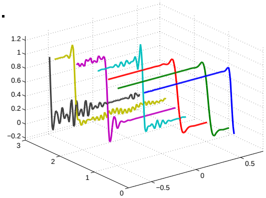

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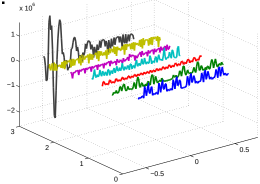

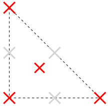

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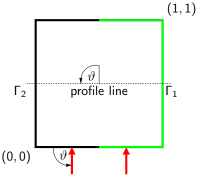

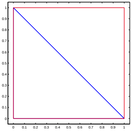

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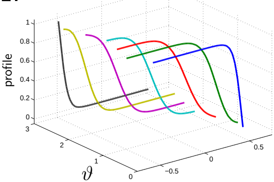

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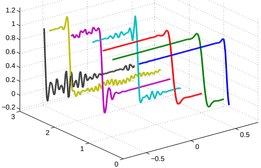

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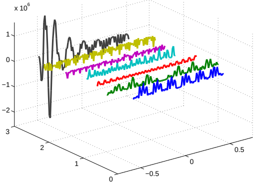

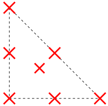

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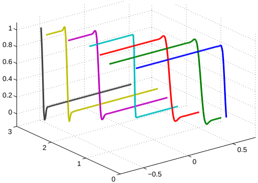
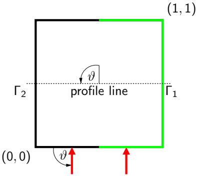
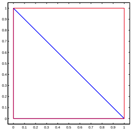
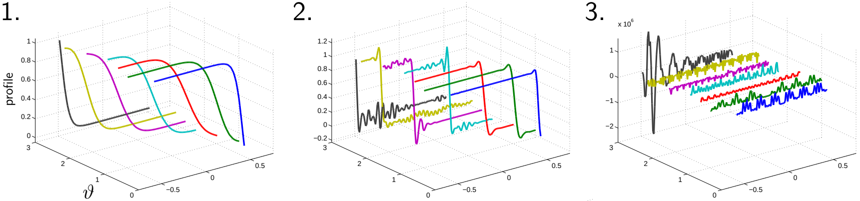


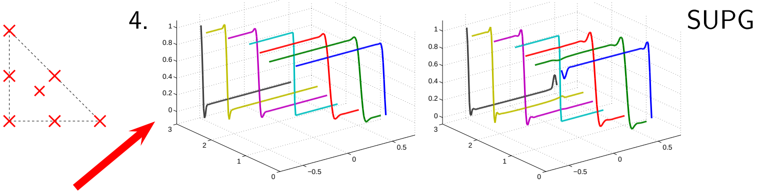
Numerical Experiments: Smoothing

- $\beta_1 = cos(\vartheta)$, $\beta_2 = sin(\vartheta)$, $f \equiv 0$
- $u \equiv 1$ on Γ_1, $u \equiv 0$ on Γ_2
- $\varepsilon = 10^{-14}$, $h = 0.042$









$\Rightarrow$ Quadrature rule using vertices, midpoints and barycenter competes with SUPG!

Conclusions and Further Issues

- Lie derivative formalism reproduces upwind FEM!
- Can be extended to higher order!
- Choice of basis and quadrature?
- Proof of stability for 2nd+ order?
- Lie Derivative formalism brings Upwinding to discretization!

$$\boldsymbol{\beta} \times \mathbf{curl A} \simeq i_{\boldsymbol{\beta}} d\omega_1$$

- Stability of discretizations for 1+ forms, e.g. magnetic convection?
- Boundary and gauge conditions for 1+ forms?