Packings in Complete Graphs

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Abstract

We deal with the concept of packings in graphs, which may be regarded as a generalization of the theory of graph design. In particular we construct a vertex- and edge-disjoint packing of $K_n$ (where $\frac{n}{2} \mod 4$ equals 0 or 1) with edges of different cyclic length. Moreover we consider edge-disjoint packings in complete graphs with uniform linear forests (and the resulting packings have special additional properties). Further we give a relationship between finite geometries and certain packings which suggests interesting questions.

1 Introduction

In geometry the concept of packing may be described as follows: Given a closed set $A \subset \mathbb{R}^n$ and a family $\{B_i\}_{i \in \Lambda}$ of closed subsets of $A$, e.g. $A = \mathbb{R}^2$ and $B_{x,r} = \{y \in \mathbb{R}^2 : |x-y| \leq r\}$, $(x, r) \in \mathbb{R}^2 \times \mathbb{R}_+$. A packing in $A$ by the family $\{B_i\}_{i \in \Lambda}$ is an almost disjoint subset $\{B_i\}_{i \in \lambda} \subset \{B_i\}_{i \in \Lambda}$, i.e. $B_i \cap B_j$ is a zero-set in $\mathbb{R}^n$ for $i, j \in \lambda$, $i \neq j$. The density $\sigma_{\lambda}$ of a packing is defined by $\sigma_{\lambda} = \frac{1}{\mu(A)} \sum_{i \in \lambda} \mu(B_i)$ if $A$ has finite volume $\mu(A)$ and else $\sigma_{\lambda} = \lim_{j \to \infty} \frac{1}{\mu(A_j)} \sum_{i \in \lambda} \mu(B_i \cap A_j)$, where the family $\{A_j\}_{j \in \mathbb{N}}$ of subsets of $A$ of finite measure is exhausting $A$ in a regular way. The typical question is to ask for the densest packing under eventual some restrictions on the admissible subset $\{B_i\}_{i \in \Lambda}$: e.g. the densest packing in the plane $\mathbb{R}^2$ by circles of radius 1 (see [9]) or the densest packing in the unit square by ten circles of equal radius (see [7]).

It is known, that the concept of geometric packing has discrete analogues (see [10]). Here we deal with packings in (finite) graphs: Given a (finite) graph $G = (V, E)$, $V$ the set of vertices and $E$ the set of edges, and a family $\{B_i\}_{i \in \Lambda}$ of partial subgraphs $B_i = (V_i, E_i)$ of $G$. A packing in $G$ by the family $\{B_i\}_{i \in \Lambda}$ is a subset $\{B_i\}_{i \in \lambda} \subset \{B_i\}_{i \in \Lambda}$ such that either the condition

(C1) $B_i \cap B_j \subset V$ for $i, j \in \lambda$, $i \neq j$

or the condition

(C2) $B_i \cap B_j = \emptyset$ for $i, j \in \lambda$, $i \neq j$

holds. If, in the (C1)-case, the packing $\{B_i\}_{i \in \lambda}$ in $(V, E)$ has the additional property that there exists an $m \in \mathbb{N}$ such that every pair $x_i, x_k$ of distinct vertices of $V$ occurs for $m$ or $m + 1$ indices $i \in \lambda$ in a connected component of $B_i$, then we call it homogeneous (C1)*-packing. So, homogeneous (C1)*-packings are particularly regular or “well-balanced” (C1)-packings. This will become more clear in the examples we consider below.
There is always a good chance to find in the set of (C1)-packings of maximal cardinality a (C1)*-representative. The number $m$ is determined by a diophantic equation and also the number of pairs of vertices occurring $m + 1$ times in a connected component of $B_i$ (this number may happen to be zero).

Now we may ask for the optimal packing in the sense that the density $\sigma_\lambda = \frac{\text{card}(\bigcup_{i \in \lambda} E_i)}{\text{card}(E)}$ is maximal under eventual some restrictions on the admissible subset $\{B_i\}_{i \in \lambda}$.

In the words of graph design we have the following:

A (C1)-packing of a complete graph with density $\sigma_\lambda = 1$ such that all the $B_i$’s are isomorphic to a given graph $G$ is a $G$-design. A (C1)-packing of a complete graph with density $\sigma_\lambda = 1$ such that all the $B_i$’s are isomorphic to a complete graph may be regarded as a balanced incomplete block design. Further a (C2)-packing with $\sigma_\lambda = 1$ such that all the $B_i$’s are isomorphic to a complete graph on 2 vertices is a 1-factor. (For the definitions see [6].) In this sense, our concept of packings is more general than graph design.

2 Notations and Definitions

We use the standard notation of [1].

Let $K_n$ denote the complete, simple graph on $n$ vertices.

A tree $T$ is called a linear tree, if each vertex of $T$ has degree 1 or 2.

The length of a linear tree $T = (V_T, E_T)$ is the cardinality of $V_T$.

A linear forest is a set of linear trees satisfying condition (C2).

A uniform forest $F$ is a linear forest such that all linear trees of $F$ have the same length, the height of the forest.

The size of a forest $F$ is the cardinality of $F$.

Given a complete graph $K_n = (V_n, E_n)$ and $h > 1$ a divisor of $n$. Let $B_{n,h}$ denote the family

$$B_{n,h} := \{B_i = (V_i, E_i) : B_i \text{ a uniform forest of height } h \text{ and size } \frac{n}{h}\}$$

of subgraphs of $K_n$. We are interested in packings $A_{n,h} \subset B_{n,h}$ in $K_n$ by the family $B_{n,h}$ such that condition (C1) or (C1)* (as in Section 4) or condition (C2) and some additional restrictions hold (as in Section 3). In the language of graph design, a (C1)-packing $A_{n,h} \subset B_{n,h}$ in $K_n$ with density $\sigma_\lambda = 1$ is a resolvable, balanced path design (cf. [6]). In the (C1)-case it is easy to see that for a packing of $K_n$ by $B_{n,h}$ there holds

$$\text{card}(\lambda) \leq \frac{n(n - 1)}{2} \frac{2}{h - 1}$$

and because $\text{card}(\lambda)$ is an integer we get

$$\text{card}(\lambda) \leq \left\lfloor \frac{h(n - 1)}{2(h - 1)} \right\rfloor \quad (2)$$

(where $\lfloor x \rfloor$ is the nearest integer less or equal than $x$).

On the other hand if we consider packings which respect (C2) we trivially have $\text{card}(\lambda) \leq 1$: So here the question is whether a packing exists or not.
3 Packings in complete graphs by edges of different length

Let $K_n$ be the complete graph with vertices $\{x_i\}_{1 \leq i \leq n}$. We define the cyclic length of an edge $[x_i, x_j]$ joining $x_i$ and $x_j$ as

$$l([x_i, x_j]) := \min\{|i - j|, n - |i - j|\}$$

See also Figure 1 for the geometric meaning of the cyclic length. Then there holds

**Theorem 1** If $n$ is even then there exists a (C2)-packing in $K_n$ by the family $B_{n,2}$ such that only edges of different cyclic length occur, if and only if $\frac{n}{2} \mod 4$ equals 0 or 1.

**Remark 1:** If $n$ is odd the corresponding problem is trivial.

**Proof:** (i) Consider a (C2)-packing in $K_{2m}$ by $B_{2m,2}$ such that every cyclic length $1, 2, \ldots, m$ occurs. Let $P := \{x_i : i \text{ odd}\} \subset V_{2m}$ and $Q := \{x_i : i \text{ even}\} \subset V_{2m}$. If an edge of the packing has odd cyclic length it is joining the sets $P$ and $Q$, else it is joining two vertices of $P$ or of $Q$. Hence the number of edges of the packing having even cyclic length must be even. Now, if $m$ is even the even cyclic lengths occurring in the packing are $\{2, 4, \ldots, m\}$ and this set is even if and only if $m \equiv 0 \pmod{4}$. If on the other hand $m$ is odd the even cyclic lengths occurring in the packing are $\{2, 4, \ldots, m - 1\}$ and this set is even if and only if $m \equiv 1 \pmod{4}$.

(ii) For the other direction we consider two cases.

**Case 1.** $m \equiv 0 \pmod{4}$:
If $m = 4$ then $A_{8,2} := \{[x_1, x_8], [x_2, x_5], [x_3, x_7], [x_4, x_6]\}$ is a packing in $K_{2m}$ such that every cyclic length $1, 2, \ldots, m$ occurs (see Figure 1).

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**Figure 1:** (C2)-packing in $K_8$ by edges such that every cyclic length occurs.

**Figure 2:** (C2)-packing in $K_{32}$ by edges such that every cyclic length occurs.
If \( m = 4k \) \((k > 1)\) then it is easy to check that

\[
A_{2m,2} := \left\{ [x_1, x_{2k}], [x_2, x_{4k+1}], [x_{7k+2}, x_{7k+1}], \left\{ [x_i, x_{8k+1-i}] \right\}_{k < i < 2k}, \right. \\
\left. \left\{ [x_i, x_{8k+2-i}] \right\}_{2k < i \leq 4k}, \left\{ [x_i, x_{8k+3-i}] \right\}_{3 \leq i \leq k} \right\}
\]

is a packing in \( K_{2m} \) with the desired properties. Figure 2 shows the resulting packing for \( n = 32 \).

**Case 2.** \( m \equiv 1 \) (mod 4):

If \( m = 1 \) then \( A_{2,2} := \{[x_1, x_2]\} \) is a packing in \( K_{2m} \) such that the cyclic length 1 occurs.

If \( m = 5 \) then \( A_{10,2} := \{[x_1, x_2], [x_3, x_9], [x_4, x_7], [x_5, x_{10}], [x_6, x_8]\} \) is a packing in \( K_{2m} \) such that every cyclic length 1, 2, \ldots, m occurs (see Figure 3).

If \( m = 4k + 1 \) \((k > 1)\) then it is easy to check that

\[
A_{2m,2} := \left\{ [x_1, x_{4k+1}], [x_{2k}, x_{4k+2}], [x_{7k+2}, x_{7k+1}], \left\{ [x_i, x_{8k+2-i}] \right\}_{k + 2 < i < 2k}, \right. \\
\left. \left\{ [x_i, x_{8k+3-i}] \right\}_{2k < i \leq 4k}, \left\{ [x_i, x_{8k+4-i}] \right\}_{2 \leq i \leq k+1} \right\}
\]

is a packing in \( K_{2m} \) with the desired properties. Figure 4 shows the resulting packing for \( n = 34 \).

**Remark 2:** Although it was quite hard to find a packing in a complete graph by edges of different cyclic length, there exist in fact many solutions for large \( m \):

\[
K_2 : 1 \text{ solution} \\
K_8 : 1 \text{ solution} \\
K_{10} : 2 \text{ solutions} \\
K_{16} : 128 \text{ solutions} \\
\ldots : \ldots
\]
Of course, congruent solutions are identified.

**Remark 3:** These packings are in fact very special 1-factorizations of $K_{2m}$. Note that in general 1-factorizations of $K_{2m}$ always exist (cf. [4] p. 85).

## 4 High, large and balanced forests

In this section we will consider (C1) and (C1)$^*$-packings in $K_n$ by the family $B_{n,h}$. We are interested in the cases $h = n$ (hence the corresponding forests are of maximal possible height), $2h = n$ (the corresponding forests contain exactly two trees), $h = 2$ (the corresponding forests are as large as possible) and $h^2 = n$ (the corresponding forests are as large as high). We show in most of the mentioned cases that estimate (2) is sharp.

**Notation:** If $\sigma$ is a permutation of the set $\{1,\ldots,n\}$ and $H = (V_H,E_H)$ a partial subgraph of $K_n$, then $\sigma[H] = (V_{\sigma[H]},E_{\sigma[H]})$ where $V_{\sigma[H]} := \{x_{\sigma(i)} : x_i \in V_H\}$ and $E_{\sigma[H]} := \{[x_{\sigma(i)},x_{\sigma(j)}] : [x_i,x_j] \in E_H\}$ (see also Figure 5). Further let $\sigma^0$ be the identity and $\sigma^{n+1} := \sigma(\sigma^n)$.

### 4.1 High forests: $h = n$

For $h = n > 1$ we obtain by estimate (2) that a maximal packing is of cardinality less or equal than $\lfloor \frac{n}{2} \rfloor$. And indeed we find:

**Theorem 2** In $K_n$ there exists a (C1)$^*$-packing $A_{n,n}$ by $B_{n,n}$ of cardinality $\lfloor \frac{n}{2} \rfloor$.

**Proof:** Let

$$A := \{[x_1,x_n],[x_1,x_{n-1}],[x_2,x_{n-1}],[x_2,x_{n-2}],\ldots,[x_{\lfloor \frac{n}{2} \rfloor},x_{\lfloor \frac{n}{2} \rfloor+1}]\}$$

and

$$\sigma := \begin{pmatrix} 1 & 2 & 3 & \ldots & i & \ldots & n \\ 2 & 3 & 4 & \ldots & i+1 & \ldots & 1 \end{pmatrix}.$$ 

Then $A_{n,n} := \{B_i : B_i = \sigma^i[A], \ 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ is a (C1)-packing of cardinality $\lfloor \frac{n}{2} \rfloor$ (see Figure 5). Because all pairs of vertices $x_k,x_l$ belong to every $B_i \in A_{n,n}$ and since every $B_i$ is connected, the packing is trivially (C1)$^*$. □

In fact Theorem 2 follows also from [4] p. 89.

**Remark 4:** If $n$ is even, the density of the packing constructed above is 1. Hence, it can be regarded as a path design (in contrast to the case $n$ odd). At this stage we get, as a byproduct which will be useful afterwards, also an optimal (C1)$^*$-packing in $K_{n+1}$ by cycles of length $n+1$: Just introduce a new point $x_{n+1}$ and close every tree constructed above by joining both ends with $x_{n+1}$ (see Figure 6). The cardinality of this packing is $\lfloor \frac{n}{2} \rfloor$, thus it is optimal. If $n$ is even its density is 1 and hence we get a 2-factorization of $K_{n+1}$ (see [4] p. 89).
If \( n = 2k \) and if we consider each linear forest occurring in the packing \( A_{n,n} \) (constructed in the proof of Theorem 2) as a row of a matrix, we get a \( k \times n \)-matrix which yields in an natural way a horizontally complete \( k \times n \) latin rectangle (cf. [3]).

4.2 The case \( 2h = n \)

The second highest forests appear if \( 2h = n > 2 \). In this case estimate (2) says, that a maximal packing is of cardinality less or equal than \( \left\lfloor \frac{h(2h-1)}{2h-2} \right\rfloor \) which is \( h \) (\( = \frac{n}{2} \)) for \( h > 2 \).

We find:

**Theorem 3** In \( K_n \) (with \( n = 2h \)) there exists a \((C1)\)-packing \( A_{2h,h} \) by \( B_{2h,h} \) of cardinality \( \left\lfloor \frac{h(2h-1)}{2h-2} \right\rfloor \) (and hence this packing is optimal), whereas a \((C1)\)*-packing of this cardinality only exist for \( h = 2 \).

**Proof:** The case \( h = 2 \) is trivial, so let us assume \( h > 2 \). By Section 4.1 we can find a packing for \( h' = n \) (\( = 2h \)) of cardinality \( \frac{n}{2} \) (\( = h \)). Canceling an edge of each linear tree of this packing such that both parts are of length \( h \) we get a packing \( A_{2h,h} \) of cardinality \( h \).

Thus (2) is sharp also in case \( 2h = n \).

To see that for \( h > 2 \) no \((C1)\)*-packing of the mentioned cardinality exists we proceed by contradiction. Suppose there is such a packing \( A_{2h,h} = \{B_i \in B_{2h,h} : i = 1, \ldots, h\} \). Consider the sets \( S_i = \{j : x_i \text{ and } x_{2h} \text{ are in the same connected component of } B_j\} \) for \( i = 1, \ldots, 2h - 1 \). Since \( A_{2h,h} \) is a \((C1)\)*-packing the sets \( S_i \) are all of “almost equal size” or more precisely there exists \( m \in \mathbb{N} \) such that every set \( S_i \) has cardinality \( m \) or \( m + 1 \), say \( |S_1| = \ldots = |S_x| = m \) and \( |S_{x+1}| = \ldots = |S_{2h-1}| = m + 1 \). By counting edges we obtain:

\[
m = \begin{cases} 
\frac{h}{2} - 1 & \text{if } h \text{ even} \\
\frac{h-1}{2} & \text{if } h \text{ odd}
\end{cases} \quad x = \begin{cases} 
\frac{h}{2} & \text{if } h \text{ even} \\
\frac{3h-1}{2} & \text{if } h \text{ odd}
\end{cases}
\]
To continue we have to distinguish the four cases $h \equiv \iota \mod 4$, $\iota = 0, 1, 2, 3$. We only carry out $\iota = 1$ (the other cases are similar). For $h = 4k + 1$ we obtain that $|S_i \cap S_j| = k$ for $j = 2, \ldots, x$. It follows that $x_1$ and $x_j$, $j = 2, \ldots, x$, are $m + 1$ times in the same connected component of a $V_j$. But since $x - 1 = \frac{3h - 1}{2} - 1 > \frac{h - 1}{2} = 2h - 1 - x$ this is impossible. (If $\iota = 3$, consider $S_1$ and $S_j$ for $j = x + 1, \ldots, 2h - 1$.)

An alternative proof is based upon the observation that the $(C1)^*$-packing considered above would induce a partition of the set $\{1, \ldots, h\}$ into $x$ subsets $S_i$ of cardinality $m$ having the property that their intersection is of cardinality $k$. It is quite easy to see that there is no such partition.

\section{4.3 Large forests: $h = 2$}

If $h = 2$, then because $h$ is a divisor of $n$, $n$ has to be even and of the form $n = 2m$ (for an $m > 0$). Estimate (2) says, that in this case a maximal packing is of cardinality less or equal than $2^{(n-1)} = n - 1$. In fact there holds:

\begin{theorem}
If $n$ is even then there exists a $(C1)^*$-packing $A_{n, 2}$ in $K_n$ of cardinality $n - 1$.
\end{theorem}

\begin{proof}
Let $n = 2m$. We consider two cases.

\paragraph{Case 1} $m$ is odd, hence of the form $m = 2k + 1$:
Let $A_1 := \{[x_1, x_2], [x_3, x_4], \ldots, [x_{n-1}, x_n]\}$ and $\sigma_1 := \begin{pmatrix} 2 & 4 & \ldots & 2i & \ldots & 2m \\ 4 & 6 & \ldots & 2i + 2 & \ldots & 2 \end{pmatrix}$,

further $A_2 := \{[x_1, x_n]\} \cup \{[x_2, x_{n-2}], [x_3, x_{n-1}], [x_4, x_{n-4}], [x_5, x_{n-3}], \ldots, [x_m, x_{m+2}]\}$ and $\sigma_2 := \begin{pmatrix} 1 & 2 & \ldots & i & \ldots & n - 1 & n \\ 3 & 4 & \ldots & i + 2 & \ldots & 1 & 2 \end{pmatrix}$.

Then

$$A_{n, 2} := \{B_1 : B_1 = \sigma_1^{-1}[A_1] \text{ for } 1 \leq i \leq 2k \text{ and } B_1 = \sigma_2^{-2k-1}[A_2] \text{ for } 2k < i < n\}$$

is a $(C1)$-packing of cardinality $n - 1$.

\paragraph{Case 2} $m$ is even, hence of the form $m = 2k$. Here we give the proof by induction on $k$. Let $P := \{x_i : i \text{ is odd}\}$ and $Q := \{x_i : i \text{ is even}\}$. By induction there are packings $A_{2k, 2}^P = \{A_i^P : 1 \leq i < m\}$ and $A_{2k, 2}^Q = \{A_i^Q : m \leq i < n - 1\}$ in $P$ (respectively $Q$) both of cardinality $m - 1$.

Then with $A := \{[x_1, x_2], [x_3, x_4], \ldots, [x_{n-1}, x_n]\}$ and

$$\sigma := \begin{pmatrix} 2 & 4 & \ldots & 2i & \ldots & 2k \\ 4 & 6 & \ldots & 2i + 2 & \ldots & 2 \end{pmatrix},$$

define

$$A_{2m, 2} := \{B_i : B_i = \sigma^i[A] \text{ for } 0 \leq i < m \text{ and } B_i = A_i^P \cup A_i^Q \text{ for } m \leq i < n - 1\}$$

which is a $(C1)$-packing of cardinality $n - 1$. 

In both cases, the packing is trivially \((C1)^*\) since every pair of vertices is exactly once in the same connected component of a forest. \(\square\)

**Remark 5:** In fact we proved that if \(n\) is even, then \(K_n\) has a 1-factorization (cf. [4] Theorem 9.1).

### 4.4 Balanced forests: \(h^2 = n\)

For \(h^2 = n\) the estimate (2) says, that a maximal packing is of cardinality less or equal than \(\binom{h+1}{2}\).

**Lemma** If \(h\) is odd and \(n = h^2\), then there is a \((C1)\)-packing \(A_{n,h}\) in \(K_n\) of cardinality \(\frac{n-1}{2}\).

**Proof:** Use the Remark 4 to construct in \(K_n\) \(\frac{n-1}{2}\) many pairwise edge disjoint cycles of length \(n\). By canceling suitable edges in each cycle, we get a set of uniform edge disjoint forests of height \(h\), thus a \((C1)\)-packing of cardinality \(\frac{n-1}{2}\). \(\square\)

Note that the difference between \(\frac{n+h}{2}\) (the upper bound for the cardinality of a \((C1)\)-packing which is given by estimate (2)) and \(\frac{n-1}{2}\) is only \(\frac{h+1}{2}\), hence a \((C1)\)-packing in \(K_n\) of cardinality \(\frac{n-1}{2}\) looks almost optimal. However the next Theorem shows, that there are always \((C1)\)-packings, such that estimate (2) is sharp and that in some cases we can even find a \((C1)^*\)-packing of density 1.

**Theorem 5** For any \(h > 1\) there exists a \((C1)\)-packing \(A_{n,h}\) in \(K_n\) of cardinality \(\binom{h+1}{2}\) and hence of density 1. Moreover, if \(h\) is of the form \(h = p^m\) (where \(p\) is a prime number and \(m \in \mathbb{N}\)), there exists a \((C1)^*\)-packing \(A_{n,h}\) in \(K_n\) of the same cardinality and density.

**Proof:** The first part of the theorem, namely that there exist \((C1)\)-packings \(A_{n,h}\) in \(K_n\) of cardinality of density 1 follows quite easily from the results of [5], [6] and [2] (see also the interpretation of the packing as solution of the well-known “handcuffed prisoner problem”). Nevertheless, the packings constructed in the cited papers are not \((C1)^*\) as one easily checks (two prisoners may walk quite often in the same row whereas others only once). So, we have to show that for \(h\) being a power of a prime, a \((C1)^*\)-packing (and hence a particularly regular solution of the problem) of density 1 exists.

For even \(h\) we can give a shorter construction of a \((C1)\)-packing than in the mentioned papers, so let us start with

**Case 1.** \(h\) is an even number, hence of the form \(h = 2k\).

First we take the \((C2)\)-packing \(A_{n,n}\) of cardinality \(\frac{n^2}{2}\) constructed in the proof of Theorem 2. Now if we cancel in each linear tree all edges of cyclic length 0 \((\text{mod } h)\), we get a \((C2)\)-packing \(A_{n,h}\) of the same cardinality.

The canceled edges form \(h\) disjoint complete graphs \(\{K^i_h\}_{1 \leq i \leq h}\). Again by Theorem 2 we find a \((C2)\)-packing \(A^i_{h,h}\) of cardinality \(k\) in each such graph. Choosing one linear tree
(of length $h$) in each $A^i_{h,h}$ we get a uniform forest of height $h$ and size $h$. We repeat this procedure $k$ times and end up with the $k$ missing uniform forests: $\frac{h^2}{2} + k = \left(\frac{h+1}{2}\right)$.

**Case 2.** $h$ is of the form $h = p^m$, where $p$ is a prime number and $m \in \mathbb{N}$. We will give the proof of this case in three steps.

1st step: We identify the vertices of $K_n$ with the points $(i,j)$, $i, j \in F$, of the plane of the coordinate geometry over a Galois field $F$ with $h = p^m$ elements (as a general reference for finite geometry see [8]). In this plane we are given $h + 1$ bundles of parallels, each bundle consisting of $h$ nonintersecting straight lines. One bundle is consisting of the lines $l_{a_{1i}} = \{(i,j)\}_{j \in F}$, the other bundles are $l_{a_{2i}} = \{(j,sj + i)\}_{j \in F}$ (where $s \in F$). Each bundle of parallels may be considered as a partition of $V_n$, the vertices of $K_n$.

2nd step: It is easy to see that for any two partitions $P_1 = \{v_k^1 : 1 \leq k \leq h\}$ and $P_2 = \{v_k^2 : 1 \leq k \leq h\}$ constructed in step 1 there is a $(h \times h)$-matrix $A = a_{ij}$ such that $\{a_{ij} : i = k\} = v_k^1$ and $\{a_{ij} : j = k\} = v_k^2$. With the $h + 1$ partitions constructed in step 1 we obtain in this way $\frac{h+1}{2}$ many $(h \times h)$-matrices.

3rd step: Now we take a matrix $A = a_{ij}$ constructed in step 2 and show that it yields a packing in $K_n$ of cardinality $h$. Combining the $h$ packings given by each of the $\frac{h+1}{2}$ matrices we obtain a packing in $K_n$ of cardinality $\frac{h(h+1)}{2} = \left(\frac{h+1}{2}\right)$.

(a) First consider the $h$ linear trees $[a_{i,i}, a_{i+1,i}, a_{i+1,i-1}, a_{i+2,i-1}, \ldots, a_{i,\frac{h-1}{2}i-\frac{h-1}{2}}]$, where all indices are taken modulo $h$ and $i = 1, \ldots, h$. Those trees form a uniform forest $F$ in $K_n$ of height $h$ and size $h$.

(b) According to Theorem 2 it is—after a suitable rearrangement of the vertices—possible to construct $\frac{h-1}{2}$ linear trees of length $h$ in each row or column such that all these trees are pairwise edge-disjoint and also edge-disjoint with each linear tree belonging to the forest $F$. Therefore we get $\frac{h-1}{2}$ uniform forests of height $h$ and size $h$ coming from the rows of $A$ and the same number coming from the columns. Altogether we obtain $1 + \frac{h-1}{2} + \frac{h-1}{2} = h$ uniform forests of height $h$ and size $h$ which are by construction edge-disjoint.

Thus we get a (C1)-packing $A^i_{n,h}$ in $K_n$ of cardinality $\frac{h(h+1)}{2} = \left(\frac{h+1}{2}\right)$, which is by construction even a (C1)*-packing.

□

**Example:** To illustrate the construction above we consider the case $h = 3$.

1st step: Figure 7 shows the coordinateplane $F \times F$ for the finite field $F = F_3 = \{0, 1, 2\}$ and the bundles of parallels. We identify $x_1 \equiv 1 \equiv (\bar{0}, 2)$, $x_2 \equiv 2 \equiv (\bar{1}, 2)$ etc.
The bundles $l_0$, $l_1$, $l_2$, $l_\infty$.

**Figure 7**

2nd step: The partitions given by the bundles of parallels of step 1 give rise to the following 2 matrices having the property that each bundle occurs in exactly one of the matrices either in the rows or in the columns:

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{pmatrix}
\text{ and }
\begin{pmatrix}
3 & 5 & 7 \\
8 & 1 & 6 \\
4 & 9 & 2 \\
\end{pmatrix}
\]

The first matrix is built of $l_0$ and $l_\infty$, the second of $l_1$ and $l_2$ (other choices are also possible).

3rd step: By each of the two matrices of step 2 we construct packings in $K_9$ of cardinality 3. The combination gives the packing of cardinality 6.

(a) By the construction given in the proof we first get the two uniform forests \{[1, 4, 6], [5, 8, 7], [9, 3, 2]\} and \{[3, 8, 6], [1, 9, 4], [2, 7, 5]\}.

(b) At least we get the four uniform forests \{[2, 1, 3], [4, 5, 6], [7, 9, 8]\}, \{[1, 7, 4], [5, 2, 8], [3, 6, 9]\}, \{[5, 3, 7], [8, 1, 6], [4, 2, 9]\} and \{[3, 4, 8], [1, 5, 9], [7, 6, 2]\} where the first two come from the first matrix and the last two from the second matrix.

**Remark 6:** P. Hell and A. Rosa have shown in [5] that a (C1)-packing $\mathcal{A}_{h^2,h}$ of $K_{h^2}$ with density $\sigma_\lambda = 1$ always exists. The difference between our solution and the solution given in [5] for $h = p^m$ (where $p$ is a prime number) is, that our solution is homogeneous, i.e. if we take two arbitrary distinct vertices of $K_{h^2}$, then they appear in the same tree exactly $\frac{p^m - 1}{2}$ or $\frac{p^m + 1}{2}$ times if $p$ is odd and $\frac{p^m}{2}$ times if $p = 2$. The solution given in [5] is far away from being (C1)*. In the language of graph design we may summarize the results as follows.

**Summary:** If $n = h^2$, then there exists a resolvable balanced path design of type (n,h,1). Furthermore, if $h = 2^k$, then we can choose this resolvable balanced path design such that it is at the same time a balanced incomplete block design (the blocks being the vertices of the trees) with every pair of vertices occurring $2^{k-1}$ times in a block. If $h = p^m$, $p$ an odd prime number, then for diophantic reasons, there is no $m$ such that every pair of vertices
occurs exactly $m$ times in the same tree. Therefore, in this case, the $(C1)^*\text{-packing}$ we constructed is the most balanced solution one can think of.

We close with the following question.

Does a $(C1)^*\text{-packing}$ of $K_{36}$ by $B_{36,6}$ with density $\sigma_\lambda = 1$ exist?

References


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