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Conjugate conics and closed chains of Poncelet polygons

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Abstract

If a point x moves along a conic G then each polar of x with respect to a second conic A is tangent to one particular conic H, the conjugate of G with respect to A. In particular, if P is a Poncelet polygon, inscribed in G and circumscribed about A, then, the polygon P' whose vertices are the contact points of P on A is tangent to the conjugate H of G with respect to A. Hence P' is itself a Poncelet polygon for the pair A and H. P' is called dual to P. This process can be iterated. Astonishingly, there are very particular configurations, where this process closes after a finite number of steps, *i.e.*, the n-th dual of P is again P. We identify such configurations of closed chains of Poncelet polygons and investigate their geometric properties.

1 Conjugate conics

In order to make this presentation self-contained and to fix notation, we first describe our general setting. The interested reader will find extensive surveys about algebraic representations of conics in the real projective plane in [1] or [7].

A *projective plane* is an incidence structure $(\mathbb{P}, \mathbb{B}, \mathbb{I})$ of a set of *points* $\mathbb{P}$, a set of *lines* $\mathbb{B}$ and an *incidence relation* $\mathbb{I} \subset \mathbb{P} \times \mathbb{B}$. For $(p, g) \in \mathbb{I}$, it is custom to say that p and g are incident, that g passes trough p, or that p lies on g. The axioms of a projective plane are:

(A1) Given any two distinct points, there is exactly one line incident with both of them.

(A2) Given any two distinct lines, there is exactly one point incident with both of them.

(A3) There are four points such that no line is incident with more than two of them.

The *dual structure* $(\mathbb{B}, \mathbb{P}, \mathbb{I}^*)$ is obtained by exchanging the sets of points and lines, with the *dual incidence relation* $(g, p) \in \mathbb{I}^* :\Longleftrightarrow (p, g) \in \mathbb{I}$. (A1) turns into (A2) and vice versa if the words "points" and "lines" are exchanged. Moreover, one can prove that (A3) also holds for the dual relation. Hence, if a statement is true in a projective plane $(\mathbb{P}, \mathbb{B}, \mathbb{I})$, then the dual of that statement which is obtained by exchanging the words "points" and "lines" is true in the dual plane $(\mathbb{B}, \mathbb{P}, \mathbb{I}^*)$. This follows since dualizing each statement in the proof in the original plane gives a proof in the dual plane.

In this paper, we mostly work in the standard model of the real projective plane. For this, we consider $\mathbb{R}^3$ and its dual space $(\mathbb{R}^3)^*$ of linear functionals on $\mathbb{R}^3$. The set of points is $\mathbb{P} = \mathbb{R}^3 \setminus \{0\}/\sim$, where $x \sim y \in \mathbb{R}^3 \setminus \{0\}$ are equivalent, if $x = \lambda y$ for some $\lambda \in \mathbb{R}$. The set of lines is $\mathbb{B} = (\mathbb{R}^3)^* \setminus \{0\}/\sim$, where $g \sim h \in (\mathbb{R}^3)^* \setminus \{0\}$ are equivalent, if $g = \lambda h$ for some $\lambda \in \mathbb{R}$. Finally, $([x], [g]) \in \mathbb{I}$ *iff* $g(x) = 0$, where we denoted equivalence classes by square brackets. In the sequel we will identify $\mathbb{R}^3$ and $(\mathbb{R}^3)^*$ by the standard inner product $\langle \cdot, \cdot \rangle$ which allows to express the incidence $([x], [g]) \in \mathbb{I}$ through the relation $\langle x, g \rangle = 0$.

As usual, a line $[g]$ can be identified by the set of points which are incident with it. Vice versa a point $[x]$ can be identified by the set of lines which pass through it. The affine plane $\mathbb{R}^2$ is embedded in the present model of the projective plane by the map

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \left[\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \right].$$

Two projective planes $(\mathbb{P}_1, \mathbb{B}_1, \mathbb{I}_1)$ and $(\mathbb{P}_2, \mathbb{B}_2, \mathbb{I}_2)$ are *isomorphic*, if there is a bijective map $\phi \times \psi : \mathbb{P}_1 \times \mathbb{B}_1 \to \mathbb{P}_2 \times \mathbb{B}_2$ such that $(p, g) \in \mathbb{I}_1 \iff (\phi(p), \psi(g)) \in \mathbb{I}_2$.

The above constructed model, the real projective plane, is *self-dual*, *i.e.*, the plane is isomorphic to its dual. Indeed, an isomorphism is given by $([x], [g]) \mapsto ([x], [g])$, since

$$([x], [g]) \in \mathbb{I} \iff ([g], [x]) \in \mathbb{I}^* \iff \langle g, x \rangle = 0 \iff \langle x, g \rangle = 0 \iff ([x], [g]) \in \mathbb{I}^*.$$

Therefore, in particular, the *principle of plane duality* holds in our model: Dualizing any theorem in a self-dual projective plane leads to another valid theorem in that plane.

Two linear maps $A_i : \mathbb{R}^3 \to \mathbb{R}^3$, $i \in \{1, 2\}$, are equivalent, $A_1 \sim A_2$, if $A_1 = \lambda A_2$ for some $\lambda \neq 0$. A *conic* in the constructed model is an equivalence class of a regular, linear, selfadjoint map $A : \mathbb{R}^3 \to \mathbb{R}^3$ with mixed signature, *i.e.*, A has eigenvalues of both signs. It is convenient to say, a matrix A *is* a conic, instead of A is a representative of a conic. We may identify a conic by the set of points $[x]$ such that $\langle x, Ax \rangle = 0$, or by the set of lines $[g]$ for which $\langle A^{-1} g, g \rangle = 0$ (see below). Notice that, in this interpretation, a conic cannot be empty: Since A has positive and negative eigenvalues, there are points $[p], [q]$ with $\langle p, Ap \rangle > 0$ and $\langle q, Aq \rangle < 0$. Hence a continuity argument guarantees the existence of points $[x]$ satisfying $\langle x, Ax \rangle = 0$.

From now on, we will only distinguish in the notation between an equivalence class and a representative if necessary.

Fact 1.1. *Let x be a point on the conic A. Then the line Ax is tangent to the conic A with contact point x.*

Proof. We show that the line Ax meets the conic A only in x. Suppose otherwise, that $y \not\sim x$ is a point on the conic, *i.e.*, $\langle y, Ay\rangle = 0$, and at the same time on the line Ax, *i.e.*, $\langle y, Ax\rangle = 0$. By assumption, we have $\langle x, Ax\rangle = 0$. Note, that $Ax \not\sim Ay$ since A is regular, and $\langle Ay, x\rangle = 0$ since A is selfadjoint. Hence x and y both are perpendicular to the plane spanned by Ax and Ay, which contradicts $y \not\sim x$. *q.e.d.*

In other words, the set of tangents of a conic A is the image of the points on the conic under the map A. And consequently, a line g is a tangent of the conic *iff* $A^{-1}g$ is a point on the conic, *i.e.*, if and only if $\langle A^{-1}g, g\rangle = 0$.

Definition 1.2. *If P is a point, the line AP is called its polar with respect to a conic A. If g is a line, the point $A^{-1}g$ is called its pole with respect to the conic A.*

Obviously, the pole of the polar of a point P is again P, and the polar of the pole of a line g is again g. Moreover:

Fact 1.3. *If the polar of point P with resepect to a conic A intersects the conic in a point x, then the tangent in x passes through P.*

Proof. For x, we have $\langle x, Ax\rangle = 0$ since x is a point on the conic, and $\langle x, AP\rangle = 0$ since x is a point on the polar of P. The tangent in x is the line Ax, and indeed, P lies on this line, since $\langle P, Ax\rangle = \langle AP, x\rangle = 0$. *q.e.d.*

The fundamental theorem in the theory of poles and polars is

Fact 1.4. *Let g be a line and P its pole with respect to a conic A. Then, for every point x on g, the polar of x passes through P. And vice versa: Let P be a point and g its polar with respect to a conic A. Then, for every line h through P, the pole of h lies on g.*

Proof. We prove the second statement, the first one is similar. The polar of P is the line $g = AP$. A line h through P satisfies $\langle P, h\rangle = 0$ and its pole is $Q = A^{-1}h$. We check, that Q lies on g: Indeed, $\langle Q, g\rangle = \langle A^{-1}h, AP\rangle = \langle AA^{-1}h, P\rangle = \langle h, P\rangle = 0$. *q.e.d.*

The next fact can be viewed as a generalization of Fact 1.4:

Theorem 1.5. *Let A and G be conics. Then, for every point x on G, the polar of x with respect to A is tangent to the conic $H = AG^{-1}A$ in the point $x' = A^{-1}Gx$. Moreover, x' is the pole of the tangent $g = Gx$ in x with respect to A.*

Proof. It is clear, that $H = AG^{-1}A$ is symmetric and regular, and by Sylvester's law of inertia, H has mixed signature. The point x on G satisfies $\langle x, Gx\rangle = 0$. Its pole with respect to A is the line $g = Ax$. This line is tangent to H *iff* $\langle H^{-1}g, g\rangle = 0$. Indeed, $\langle H^{-1}g, g\rangle = \langle (AG^{-1}A)^{-1}Ax, Ax\rangle = \langle A^{-1}Gx, Ax\rangle = \langle Gx, x\rangle = 0$. The point $x' = A^{-1}Gx$ lies on H, since $\langle x', Hx'\rangle = \langle A^{-1}Gx, AG^{-1}AA^{-1}Gx\rangle = \langle Gx, x\rangle = 0$. The tangent to H in x' is $Hx' = AG^{-1}AA^{-1}Gx = Ax$ which is indeed the polar of x with respect to A. The last statement in the theorem follows immediately. *q.e.d.*

DEFINITION 1.6. *The conic $H = AG^{-1}A$ is called the* conjugate conic *of G with respect to A.*

REMARK: THEOREM 1.5 generalizes FACT 1.4 in the following sense: If the conic G degenerates to a point P, the conjugate conic H with respect to A degenerates to the polar of P with respect to A: Indeed, let A and G be arbitrary conics, with $G_{33} = 1$, and $P = (0, 0, 1)$ a point represented by the matrix $Q = \mathrm{diag}(p, q, 0)$, $p, q > 0$. The conic $G_\lambda = \lambda G + (1-\lambda)Q$ in the pencil generated by G and Q degenerates to the point P as $\lambda \searrow 0$. Then

$$H_0 := \lim_{\lambda\searrow 0} \lambda A G_\lambda^{-1} A = \begin{pmatrix} A_{13}^2 & A_{13}A_{23} & A_{13}A_{33} \\ A_{13}A_{23} & A_{23}^2 & A_{23}A_{33} \\ A_{13}A_{33} & A_{23}A_{33} & A_{33}^2 \end{pmatrix}$$

and $0 = \langle x, H_0 x\rangle = (A_{13}x_1 + A_{23}x_2 + A_{33}x_3)^2$ agrees with the polar AP of P with respect to A.

The following facts follow directly from the definition.

- *H is conjugate to G with respect to A iff G is conjugate to H with respect to A.*
- *G is conjugate to itself with respect to G.*
- *If H is conjugate to G with respect to A, and G is conjugate to J with respect to A, then $H = J$.*

2 Chains of conjugate conics

We are now considering a sequence of conics $G_0, G_1, G_2, \ldots$ such that G_{i+1} is conjugate to G_{i-1} with respect to G_i for all $i \geq 1$. Such a sequence will be called a *sequence of conjugate conics.*

THEOREM 2.1. *$G_0, G_1, G_2, \ldots$ is a sequence of conjugate conics iff $G_{i+1} \sim G_1(G_0^{-1}G_1)^i$ for all $i \geq 1$.*

Proof. We proceed by induction. The formula for $i = 1$ is just the definition of conjugate conics. Now let us assume, that the formula is correct for some index $i \geq 1$. Then

$$\begin{aligned}
G_{i+2} &\sim G_{i+1}G_i^{-1}G_{i+1} \sim \\
&\sim G_1(G_0^{-1}G_1)^i\big(G_1(G_0^{-1}G_1)^{i-1}\big)^{-1}G_1(G_0^{-1}G_1)^i = \\
&= G_1(G_0^{-1}G_1)^i\big((G_0^{-1}G_1)^{i-1}\big)^{-1}G_1^{-1}G_1(G_0^{-1}G_1)^i = \\
&= G_1(G_0^{-1}G_1)^{i+1},
\end{aligned}$$

which is the formula for the index $i + 1$. *q.e.d.*

A sequence $G_0, G_1, G_2, \ldots$ of conjugate conics is called *closed cycle* or *chain of length* $n \geq 2$, if $G_k \sim G_{k+n}$ for all $k \geq 0$, and if n is minimal with this property. Then, THEOREM 2.1 gives immediately the following:

THEOREM 2.2. *Let I denote the 3×3 identity matrix. A sequence $G_0, G_1, G_2, \ldots$ of conjugate conics is a closed cycle of length n iff $(G_0^{-1}G_1)^n \sim I$ and if n is minimal with this property. In this case, there are representatives of the conics such that*

$$(G_0^{-1}G_1)^n = I. \tag{1}$$

Proof. The relation $(G_0^{-1}G_1)^n = \lambda I \sim I$ (for some $\lambda \neq 0$) is a direct consequence of THEOREM 2.1. By replacing G_1 by $\lambda^{-1/n}G_1$, one gets (1). Observe, that the case n even and $\lambda < 0$ does not occur, as can be seen by considering the determinant of $(G_0^{-1}G_1)^n = \lambda I$. *q.e.d.*

Let us briefly discuss the nontrivial solutions $S \in \mathbb{R}^{3\times 3}$ of the equation $S^n = I$.

LEMMA 2.3. *Let $S \in \mathbb{R}^{3\times 3}$ and $n > 2$ be a natural number. Then the following are equivalent:*

(a) S is a solution of $S^n = I$ and $S^m \neq \pm I$ for $1 \leq m < n$.

(b) $S = RBR^{-1}$ where $R \in \mathbb{R}^{3\times 3}$ is a regular matrix and

$$B = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos(2\pi\ell/n) & \sin(2\pi\ell/n) \\ 0 & -\sin(2\pi\ell/n) & \cos(2\pi\ell/n) \end{pmatrix}$$

where $\varepsilon = 1$ if n is odd, and $\varepsilon = \pm 1$ if n is even, and where $\ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ satisfies

$$(*) \begin{cases} (\ell, n) = 1 & \text{if } \varepsilon = 1, \\ (\ell, n/2) = 1 \text{ and } \ell n/2 \text{ even} & \text{if } \varepsilon = -1. \end{cases}$$

If S satisfies one of the above equivalent conditions, then S is diagonalizable, its eigenvalues are n-th roots of unity, and the minimal polynomial of S is the characteristic polynomial of S. Moreover:

- *If n is odd, then*
  - $\det(S) = 1$,
  - 1 *is an eigenvalue of S of algebraic multiplicity* 1*, and it is the only real eigenvalue.*
- *If n is even, then*
  - $\det(S) \in \{-1, 1\}$,
  - 1 *or* -1 *is an eigenvalue of S of algebraic multiplicity* 1*, and S has only one real eigenvalue.*

Finally, concerning (b), if x_1 is the eigenvector of S corresponding to the real eigenvalue ε, and x_2 the eigenvector to the complex eigenvalue $\mathrm{e}^{2\pi i\ell/n}$ with positive imaginary part, then the transformation matrix R has the columns $x_1, \operatorname{Re} x_2, \operatorname{Im} x_2$.

Proof. We first assume (a) and show (b): Observe, that two similar matrices annihilate the same polynomials. Assume that S has a non-diagonal Jordan decomposition $S = TJT^{-1}$, then J would also solve $J^n = I$, which clearly is not possible. Hence, S is diagonalizable and $S = TJT^{-1}$ for a regular matrix $T \in \mathbb{C}^{3\times 3}$ and $J = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $J^n = I$. Hence, the eigenvalues $\lambda_m \in \mathbb{C}$ are n-th roots of unity and the columns of T are the corresponding eigenvectors x_1, x_2, x_3. The characteristic polynomial $p_S(\lambda) = \det(S - \lambda I)$ has real coefficients and is of degree 3. Hence, S has at least one real eigenvalue $\lambda_1 = \varepsilon$, with $\varepsilon = 1$ if n is odd, and $\varepsilon \in \{-1, 1\}$ if n is even. On the other hand, if λ_2 is also real, it follows that λ_3 is real as well, and thus $J^2 = I$ which implies $S^2 = I$ contradicting the assumption in (a). It follows that $\lambda_2 = \bar{\lambda}_3 \notin \mathbb{R}$ and, without loss of generality, $\lambda_2 = e^{2\pi i\ell/n}$ for some $\ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ has a positive imaginary part.

Since $B^m \neq \pm I$ for $1 \leq m < n$, we get the following conditions for ℓ: If $\varepsilon = 1$, then, for $1 \leq m < n$, $m \cdot \ell$ is not a multiple of n (otherwise, $B^m = I$), which implies $(\ell, n) = 1$. If $\varepsilon = -1$ (which implies that n is even), then, for $1 \leq m < n$ and m even, $m \cdot \ell$ is not an even multiple of $n/2$ (otherwise, $B^m = I$), and for $1 \leq m < n$ and m odd, $m \cdot \ell$ is not an odd multiple of $n/2$ (otherwise, $B^m = -I$). In other words, if $\varepsilon = -1$, then, for any integer m with $1 \leq m < n$, if s is a solution for the equation $m \cdot \ell = s \cdot n/2$, then s and m must have different parity. This shows that if $n/2$ is even then ℓ must be odd (otherwise, for $m = n/2$ and $s = \ell$ we get $m \cdot \ell = s \cdot n/2$ where m and s are both even). Similarly, we get that if $n/2$ is odd then ℓ must be even. Let us now assume that $n/2$ is even and that ℓ is odd. If $m \cdot \ell = s \cdot n/2$, where $m \neq n/2$, then $(m \pm n/2) \cdot \ell = (s \pm \ell) \cdot n/2$. Now, since

$n/2$ was assumed to be even, $m \pm n/2$ has the same parity as m, and since ℓ is odd, s and $s \pm \ell$ have different parities. Hence, if there are integers m, s where $1 \leq m < n/2$ such that $m \cdot \ell = s \cdot n/2$, then there are integers m', s' where $1 \leq m' < n$ such that $m' \cdot \ell = s' \cdot n/2$ and m' & s' have the same parity. So, there are no m' & s' with the same parity which solve the equation $m' \cdot \ell = s' \cdot n/2$, where $1 \leq m' < n$ and $m' \neq n/2$, if and only if there are no m & s which solve the equation $m \cdot \ell = s \cdot n/2$ where $1 \leq m < n/2$. This gives us the condition $(\ell, n/2) = 1$. The case when $n/2$ is odd and consequently ℓ is even gives also $(\ell, n/2) = 1$, which shows condition $(*)$.

Now, let

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & 1/2 & i/2 \end{pmatrix}.$$

Then, $B = U^{-1}JU$, and $S = RBR^{-1}$ for $R = TU \in \mathbb{R}^{3\times3}$ with columns $x_1, \operatorname{Re} x_2, \operatorname{Im} x_2$. This shows (b).

An elementary calculation shows, that (b) implies (a).

To conclude, observe, that $S^n = I$ implies $\det(S)^n = 1$ and hence $\det(S) = 1$ if n is odd, and $\det(S) \in \{-1, 1\}$ if n is even. Finally, since the minimal polynomial and the characteristic polynomial share the same zeros, they must agree as we have three simple zeros. *q.e.d.*

The corresponding lemma for $n = 2$ is as follows:

LEMMA 2.4. *Let $S \in \mathbb{R}^{3\times3}$. Then, the following are equivalent:*

(a) S is a solution of $S^2 = I$, $S \neq \pm I$.

(b) $S = RJR^{-1}$ where $R \in \mathbb{R}^{3\times3}$ is a regular matrix, and $J = \operatorname{diag}(1, 1, -1)$ or $J = \operatorname{diag}(1, -1, -1)$.

If S satisfies one of the equivalent conditions, the minimal polynomial of S is $p(x) = x^2 - 1$.

The proof uses the same arguments as in the proof of Lemma 2.3.

2.1 Self-conjugate pairs of conics

Let us consider the case $n = 2$ in THEOREM 2.2: For $n = 2$, we obtain *self-conjugate pairs* of conics. From Lemma 2.4, we know that all solutions $S \neq \pm I$ of $S^2 = I$ are obtained in the form $S = R^{-1}JR$, where $R \in \mathbb{R}^{3\times3}$ is regular, and $J \in \mathbb{R}^{3\times3}$ is a diagonal matrix with diagonal elements in $\{-1, 1\}$ with mixed signature. (Observe, that here we exchange the rôle of R and R^{-1} compared to Lemma 2.4, because the resulting formulas turn out slightly nicer.) Then, $S = G_0^{-1} G_1$, *i.e.*,

$$G_1 = G_0 R^{-1} J R. \tag{2}$$

By a suitable projective map, we may assume without loss of generality, that $G_0 = \operatorname{diag}(1, 1, -1)$ is a circle. Observe, that we only accept solutions for G_1 in (2) which are selfadjoint and have mixed signature. We restrict ourselves to the discussion of the case $J = \operatorname{diag}(1, 1, -1)$.

It is then easy to see that $G_1 = G_1^\top$ in (2) implies

$$R_{11}R_{31} + R_{12}R_{32} - R_{13}R_{33} = R_{21}R_{31} + R_{22}R_{32} - R_{23}R_{33} = 0. \tag{3}$$

In other words, the first two rows of R are orthogonal to the third one with respect to the Minkowski inner product $x^\top J y$ induced by J. Condition (3) is actually sufficient for G_1 to be symmetric as it implies that

$$\begin{aligned} G_1 &= G_0 R^{-1} J R = \\ &= \frac{-1}{R_{31}^2 + R_{32}^2 - R_{33}^2} \begin{pmatrix} R_{31}^2 - R_{32}^2 + R_{33}^2 & 2R_{31}R_{32} & 2R_{31}R_{33} \\ 2R_{31}R_{32} & -R_{31}^2 + R_{32}^2 + R_{33}^2 & 2R_{32}R_{33} \\ 2R_{31}R_{33} & 2R_{32}R_{33} & R_{31}^2 + R_{32}^2 + R_{33}^2 \end{pmatrix} \end{aligned} \tag{4}$$

is symmetric and together with G_0 a solution of (1) for $n = 2$. The eigenvalues of G_1 are

$$1, \ -\frac{\left(R_{33} + \sqrt{R_{31}^2 + R_{32}^2}\right)^2}{R_{31}^2 + R_{32}^2 - R_{33}^2}, \ -\frac{\left(R_{33} - \sqrt{R_{31}^2 + R_{32}^2}\right)^2}{R_{31}^2 + R_{32}^2 - R_{33}^2}$$

Hence, G_1 has mixed signature iff the third row of R is spacelike with respect to the Minkowski inner product induced by J, *i.e.*, $R_{31}^2 + R_{32}^2 > R_{33}^2$. We therefore obtain:

Proposition 2.5. *Let $J = \operatorname{diag}(1, 1, -1)$, and $R \in \mathbb{R}^{3\times 3}$ be such that its first and second row are orthogonal to the third row with respect to the Minkowski inner product induced by J and the third row is spacelike. Then $G_0 = \operatorname{diag}(1, 1, -1)$ and $G_1 = G_0 R^{-1} J R$ is a closed chain of conjugate conics of length 2. In other words, the polar p of every point P on G_0 with respect to G_1 is tangent to G_0, and the polar q of every point Q on G_1 with respect to G_0 is tangent to G_1.*

Figure 1 shows an example of such a self-conjugate pair G_0, G_1 of conics.

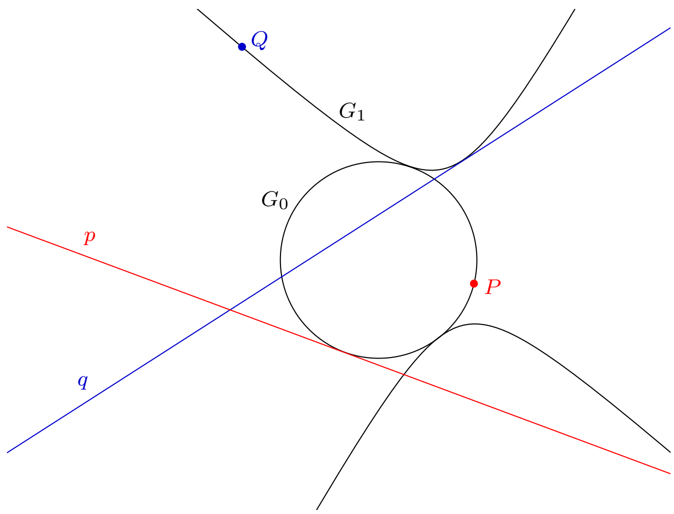

Figure 1: An example of a self-dual pair of conics: The polar p of every point P on G_0 with respect to G_1 is tangent to G_0, and the polar q of every point Q on G_1 with respect to G_0 is tangent to G_1.

Formula (4) constitutes a map $G_1 : U \subset \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ from an open set U into the 3×3 matrices: Obviously, $G_1(R_{31}, R_{32}, R_{33}) = G_1(\lambda R_{31}, \lambda R_{32}, \lambda R_{33})$ for all $\lambda \neq 0$. Hence, if we choose

$$\zeta : \mathbb{R}^2 \to \mathbb{R}^3, (\phi, \psi) \mapsto (\cosh\psi\cos\phi, \cosh\psi\sin\phi, \sinh\psi)$$

we can describe the set of solutions by composing $G_1 \circ \zeta$ to obtain

Proposition 2.6. *Let $G_0 = \operatorname{diag}(1, 1, -1)$. Then, the component of selfadjoint solutions G_1 of $(G_0^{-1}G_1)^2 = I$ with mixed signature which contains $G_1 = \operatorname{diag}(1, -1, 1)$ is a two dimensional immersed manifold in $\mathbb{R}^{3\times 3}$, parametrized by*

$$G_1(\phi, \psi) = \begin{pmatrix} \sinh^2\psi + \cos(2\phi)\cosh^2\psi & \sin(2\phi)\cosh^2\psi & \cos\phi\sinh(2\psi) \\ \sin(2\phi)\cosh^2\psi & \sinh^2\psi - \cos(2\phi)\cosh\psi^2 & \sin\phi\sinh(2\psi) \\ \cos\phi\sinh(2\psi) & \sin\phi\sinh(2\psi) & \cosh(2\psi) \end{pmatrix}$$

for $(\phi, \psi) \in \mathbb{R}^2$.

Proof. It is easy to check that the rank of the differential of this map is 2. *q.e.d.*

The case $G_0 = \operatorname{diag}(1, 1, -1)$ and $J = \operatorname{diag}(1, -1, -1)$ is similar and yields a second component of solutions via (2). Interestingly, the characteristic polynomial of the pencil generated by G_0, G_1 is an invariant, which distinguishes both cases:

LEMMA 2.7. *Let G_0, G_1 be two arbitrary different conics solving (1) for $n = 2$, i.e., $G_1 = G_0 R^{-1} J R$ for a suitable regular matrix $R \in \mathbb{R}^{3\times 3}$ and $J = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2)$, $\lambda_i \in \{-1, 1\}$, with mixed signature. Then, $\det(G_1 - \lambda G_0)$ is, up to a factor, equal to the characteristic polynomial of J.*

Proof. We have

$$\begin{aligned} \det(G_1 - \lambda G_0) &= \det(G_0 R^{-1} J R - \lambda G_0) \\ &= \det(G_0)\det(R^{-1} J R - \lambda I) \\ &= \det(G_0)\det(J - \lambda I) \end{aligned} \tag{5}$$

q.e.d.

We will encounter a similar phenomenon for values $n > 2$.

2.2 Closed chains of conjugate conics

THEOREM 2.8. *There are closed chains of conjugate conics of arbitrary length.*

The proof is constructive:

Proof. Let

$$G_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) & 0 \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} -1 & 0 & \frac{5}{4} \\ 0 & \frac{3}{4} & 0 \\ -\frac{5}{4} & 0 & 1 \end{pmatrix},$$

and $G_1 = G_0 R^{-1} J R$. Then, for $G_{i+1} = G_{i-1} G_{i-2}^{-1} G_{i-1}$, we have by induction for $i \geq 1$

$$G_i = \begin{pmatrix} \frac{1}{9}(25 - 16\cos(\frac{2i\pi}{n})) & \frac{4}{3}\sin(\frac{2i\pi}{n}) & -\frac{40}{9}\sin(\frac{i\pi}{n})^2 \\ \frac{4}{3}\sin(\frac{2i\pi}{n}) & \cos(\frac{2i\pi}{n}) & -\frac{5}{3}\sin(\frac{2i\pi}{n}) \\ -\frac{40}{9}\sin(\frac{i\pi}{n})^2 & -\frac{5}{3}\sin(\frac{2i\pi}{n}) & \frac{1}{9}(16 - 25\cos(\frac{2i\pi}{n})) \end{pmatrix}$$

and in particular, $G_n = G_0$ and $G_i \neq G_j$ for $0 \leq i \neq j < n$. G_1 has mixed signature, since the eigenvalues are

$$-1, \quad \frac{1}{18}\left(50 - 32\cos\left(\frac{2\pi}{n}\right) \pm \sqrt{\left(32\cos\left(\frac{2\pi}{n}\right) - 50\right)^2 - 324}\right). \tag{6}$$

q.e.d.

Remark: By the implicit function theorem it is easy to check that there is a differentiable function

$$\rho : U \subset \mathbb{R}^6 \to \mathbb{R}^3, \quad (R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}) \mapsto (\rho_{13}, \rho_{23}, \rho_{33})$$

on an open set U such that for G_0 and J as in the proof above, and for

$$R = \begin{pmatrix} R_{11} & R_{12} & \rho_{13} \\ R_{21} & R_{22} & \rho_{23} \\ R_{31} & R_{32} & \rho_{33} \end{pmatrix}$$

the matrix $G_1 = G_0 R^{-1} J R$ is symmetric and has mixed signature. In particular G_0, G_1 solve (1).

Figure 2 shows a closed chain of conjugate conics of length $n = 27$ as constructed in the proof of Theorem 2.8.

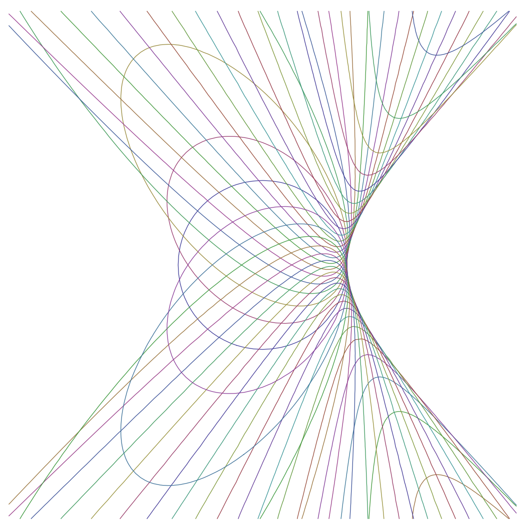

Figure 2: Closed chain of conjugate conics of length $n = 27$.

3 Closed chains of dual Poncelet polygons

In 1813, while Poncelet was in captivity as war prisoner in the Russian city of Saratov, he discovered his famous closing theorem which, in its simplest form, reads as follows (see [8]):

Theorem 3.1. *Let K and C be smooth conics in general position which neither meet nor intersect. Suppose there is an k-sided polygon inscribed in K and circumscribed about*

C. Then for any point P of K, there exists a k-sided polygon, also inscribed in K and circumscribed about C, which has P as one of its vertices.

See [3], [4] for classical overviews about Poncelet's Theorem, or [5] for a new elementary proof based only on Pascal's Theorem.

Figure 3 shows the case of two Poncelet polygons with five vertices. Observe, that the

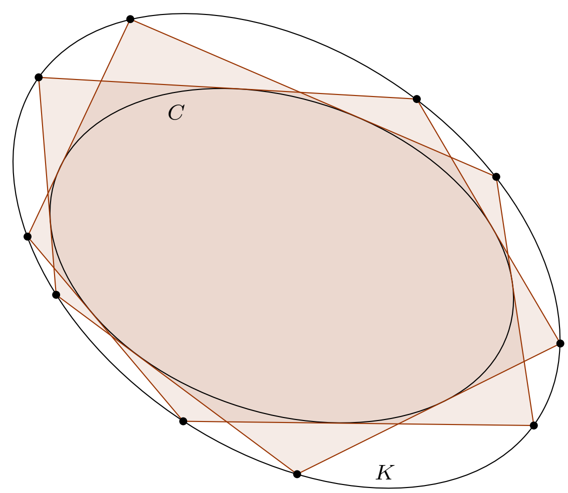

Figure 3: Two Poncelet pentagons.

polar of a vertex of a Poncelet polygon on K with respect to C joins the contact points of its ajacent sides with C. Therefore, we have:

THEOREM 3.2. *Let K and C be conics, and P a Poncelet polygon, inscribed in K and circumscribed about C. Then, the polygon whose vertices are the contact points of P on C is tangent to the conjugate conic of K with respect to C. Vice versa: The polygon formed by the tangents in the vertices of P on K has its vertices on the conjugate conic to C with respect to K.*

Two Poncelet polygons which are related in the way described in Theorem 3.2 will be called *dual*. See Figure 4 for an illustration.

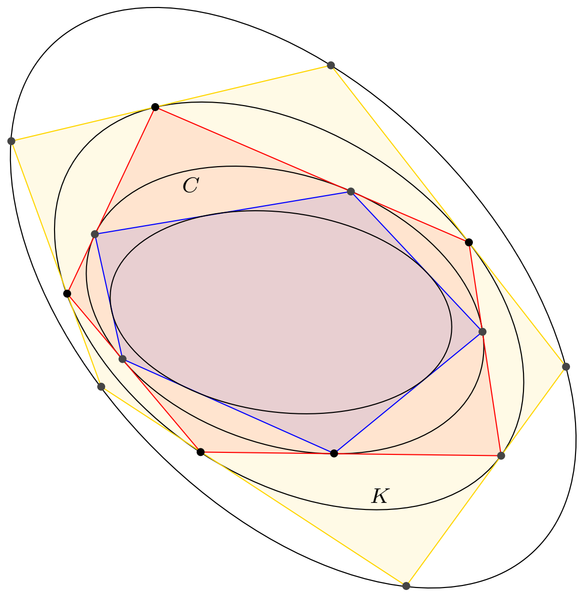

Figure 4: A Poncelet polygon (red) and its "inner" (blue) and "outer" (yellow) dual.

The process of forming dual Poncelet polygons can be continued iteratively in both directions. It has been observed in [6], that such chains of dual Poncelet polygons may close in finite projective planes. At first sight, it is counter intuitive that this phenomenon could occur in the real projective plane as well.

Let us nonetheless try to find conics G_0, G_1 which carry a Poncelet polygon, and which, at the same time, build a closed chain $G_0, G_1, \ldots, G_n = G_0$ of conjugate conics. We already know, that equation (1) must hold in order to satisfy the second condition. For the first condition, namely that G_0, G_1 carry a Poncelet k-gon, we recall the Cayley criterion (see [2]):

Theorem 3.3 (Cayley criterion). *Let G_0, G_1 be conics, $D(\lambda) = \det(G_0 + \lambda G_1)$, and*

$$\sqrt{D(\lambda)} = c_0 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3 + \ldots$$

Then, there exists a Poncelet k-gon with vertices on G_1 and tangent to G_0 if and only if

$$\det\begin{pmatrix} c_3 & c_4 & \ldots & c_{p+1} \\ c_4 & c_5 & \ldots & c_{p+2} \\ \ldots & & & \\ c_{p+1} & c_{p+2} & \ldots & c_{2p-1} \end{pmatrix} = 0 \qquad \textit{for } k = 2p,$$

or

$$\det\begin{pmatrix} c_2 & c_3 & \dots & c_{p+1} \\ c_3 & c_4 & \dots & c_{p+2} \\ \dots & & & \\ c_{p+1} & c_{p+2} & \dots & c_{2p} \end{pmatrix} = 0 \qquad \textit{for } k = 2p+1.$$

We have seen in the previous section, that the solution set of (1) has a multidimensional parameter space and consists of several connected components. For $n = 2$ it follows from Lemma 2.7, that $D(\lambda)$ is, up to a factor, constant on each of these components. It turns out, that this holds true for $n > 2$ as well:

LEMMA 3.4. *Let G_0, G_1 be two conics satisfying $(G_0^{-1}G_1)^n = I$, $n > 2$, $(G_0^{-1}G_1)^m \neq \pm I$ for $1 \leq m < n$, and $D(\lambda) = \det(G_0 + \lambda G_1)$ the characteristic polynomial of the pencil generated by G_0, G_1. Then $D(\lambda)$ is, up to a factor,*

$$(\lambda + \varepsilon)(\lambda^2 + 2\lambda\cos(2\ell\pi/n) + 1)$$

where $\varepsilon = 1$ if n is odd and $\varepsilon = \pm 1$ if n is even, and where $\ell \in \{1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$ satisfies

$$\begin{cases} (\ell, n) = 1 & \textit{if } \varepsilon = 1 \\ (\ell, n/2) = 1 \textit{ and } \ell n/2 \textit{ even} & \textit{if } \varepsilon = -1. \end{cases}$$

Proof. By Lemma 2.3, we have $G_1 = G_0 R B R^{-1}$ for a regular matrix $R \in \mathbb{R}^{3\times 3}$ and

$$B = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \cos(2\pi\ell/n) & \sin(2\pi\ell/n) \\ 0 & -\sin(2\pi\ell/n) & \cos(2\pi\ell/n) \end{pmatrix}$$

with ε and ℓ as specified above. Hence,

$$\begin{aligned} D(\lambda) &= \det(G_0 + \lambda G_1) \\ &= \det(G_0 + \lambda G_0 R B R^{-1}) \\ &= \det(G_0)\det(I + \lambda R B R^{-1}) \\ &= \det(G_0)\det(I + \lambda B) \\ &= \det(G_0)(1 + \lambda\varepsilon)(1 + 2\lambda\cos(2\ell\pi/n) + \lambda^2) \end{aligned}$$

which completes the proof. *q.e.d.*

The converse is also true:

LEMMA 3.5. *Let $G_0 \neq G_1$ be two conics and $D(\lambda) = \det(G_0 + \lambda G_1)$. Suppose that $D(\lambda)$ is, up to a factor, of the form*

$$(\lambda + \varepsilon)(\lambda^2 + 2\lambda\cos(2\ell\pi/n) + 1)$$

where $\varepsilon = 1$ *if* n *is odd and* $\varepsilon = \pm 1$ *if* n *is even, and where* $\ell \in \{1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ *satisfies*

$$\begin{cases} (\ell, n) = 1 & \text{if } \varepsilon = 1 \\ (\ell, n/2) = 1 \text{ and } \ell n/2 \text{ even} & \text{if } \varepsilon = -1. \end{cases}$$

Then, $(G_0^{-1}G_1)^n = I$*, and* $(G_0^{-1}G_1)^m \neq \pm I$ *for* $1 \leq m < n$*.*

Proof. The characteristic polynomial of $G_0^{-1}G_1$ is

$$\begin{aligned} \det(G_0^{-1}G_1 - \lambda I) &= \det(G_0^{-1}) \det(G_1 - \lambda G_0) \\ &= \det(G_0^{-1})(-\lambda)^3 \det(G_0 + \frac{1}{-\lambda} G_1) \\ &= \det(G_0^{-1})(-\lambda)^3 D(-\frac{1}{\lambda}) \\ &= (1 - \lambda\varepsilon)(1 - 2\lambda \cos(2\pi\ell/n) + \lambda^2) \end{aligned}$$

up to a factor. The roots are ε and $e^{\pm 2\pi i \ell/n}$. Hence, the characteristic polynomial of $G_0^{-1}G_1$ is a factor of $V(\lambda) = \lambda^n - 1$. Thus, the claim follows by the Cayley-Hamilton Theorem. *q.e.d.*

The question is now, for which n (the length of a cycle of conjugate conics starting with G_0, G_1), ℓ (the parameter in $D(\lambda)$ in the Lemmas 3.4 and 3.5) and k (the length of the Poncelet polynomial), the Cayley criterion is satisfied. In the following theorem we consider the case $n = 3$.

Theorem 3.6. *Each closed chain of conjugate conics of length* $n = 3$ *carries closed Poncelet triangles. Moreover, the third dual of the first Poncelet triangle is again the first Poncelet triangle.*

Remark: Observe, that $(G_0^{-1}G_1)^3 = I$ is equivalent to $(G_1^{-1}G_0)^3 = I$. In particular, no matter whether we start with a Poncelet triangle with vertices on G_0 which is tangential to G_1 or the other way round, we always get a closed cycle of dual Poncelet triangles.

Proof of Theorem 3.6. By the Lemma 3.4, we have that $D(\lambda) = 1 + \lambda^3$ and

$$\sqrt{D(\lambda)} = 1 + \frac{x^3}{2} - \frac{x^6}{8} + \frac{x^9}{16} - \frac{5x^{12}}{128} + \ldots$$

up to a factor. For $p = 1$ and $k = 2p + 1 = 3$, the determinant in the Cayley criterion is $c_2 = 0$, the coefficient of x^2, which shows the first part of the theorem.

For the second part, let G_0, G_1, G_2 be a closed chain of conjugate conics and let $\Delta_0, \Delta_1, \Delta_2$ be three Poncelet triangles such that Δ_i has its vertices on G_i and the vertices of Δ_{i+1} are the contact points of Δ_i, where we take indices modulo 3. Let A, B, C be the vertices of Δ_0. By a projective transformation we may assume that $C = (0, 0, 1)$, and that the contact points of AC, BC, AB are $(1, 1, 0), (-1, 1, 0), (0, -1, 1)$ respectively. Recall that any four point, where no three of them are collinear, can be mapped by a projective transformation to any four points, where no three of them are collinear. Thus, $(1, 1, 0), (-1, 1, 0), (0, -1, 1)$ are the vertices of Δ_1. Now, since a conic is uniquely defined by two tangents with their contact points and an additional point, we get that G_1 is a hyperbola. Moreover, G_1 is the hyperbola $x^2 - y^2 + z^2 = 0$, which implies that $A = (-1, -1, 1)$ and $B = (1, -1, 0)$. Let P, Q, R be the vertices of Δ_2, where P is the contact point of the line $(-1, 1, 0) - (1, 1, 0)$, Q the contact point of $(0, -1, 1) - (-1, 1, 0)$, and R that of $(0, -1, 1) - (1, 1, 0)$. In particular we get that G_2 has just one point at infinity, namely P, which shows that G_2 is a parabola. Since P is a point at infinity, we get that the two tangents PQ and PR to G_0 are parallel. Since the parabola is uniquely defined by the three tangents $(-1, 1, 0) - (1, 1, 0)$, $(0, -1, 1) - (-1, 1, 0)$, $(0, -1, 1) - (1, 1, 0)$ and the two contact points P and Q, the contact point R on $(0, -1, 1) - (1, 1, 0)$ is determined. Moreover, by an easy calculation we get that $\overline{AA'} = \overline{BB'}$, where A' and B' are the intersection points of AB with PQ and PR respectively.

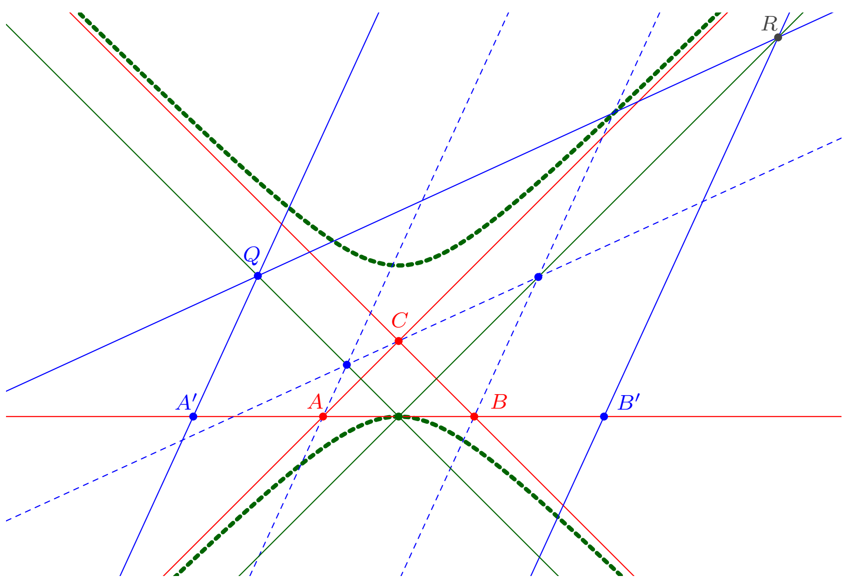

Figure 5: The situation when $A \neq A'$.

If $A = A'$, then $B = B'$ and QR is tangent to G_0 with contact point C, which shows that the third dual of Δ_0 is again Δ_0.

Otherwise, G_0 is a conic containing A, B, C where PQ, QR, RP are tangents. In general, there are four conics going through three given points and having two given tangents. So, there are four conics going through A, B, C with tangents PQ and PR.

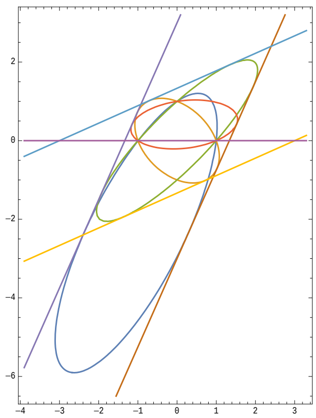

Figure 6: The four ellipses going through A, B, C with tangents PQ and PR.

However, there are just two conics going through the three points A, B, C with the three tangents PQ, QR, PR. In fact, the two conics turn out to be two ellipses, both with center $Z = (0, -1, 1)$, where PQ, QR, RP are three sides of a rhombus which is tangential to G_0.

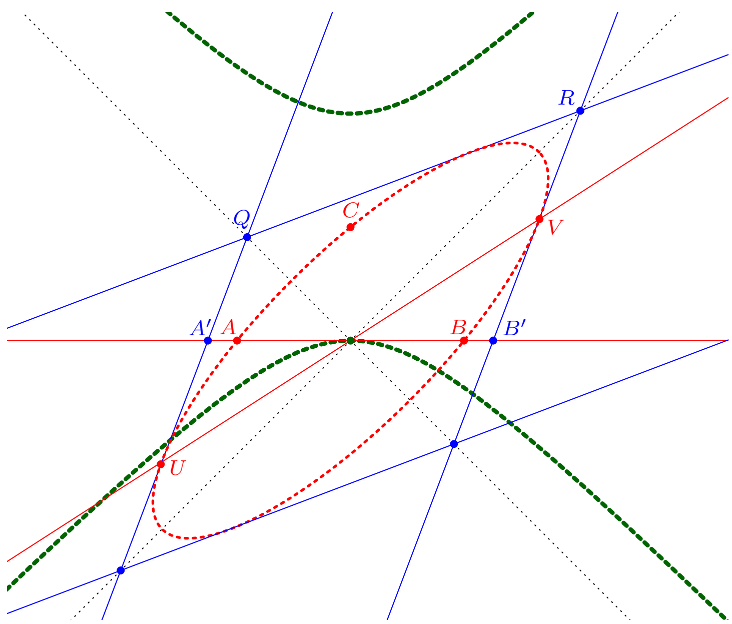

Figure 7: One of the two ellipses going through A, B, C with tangents PQ, PR, QR.

Let U and V be the contact points of PQ and PR with one of these ellipses. Then, UV goes trough Z and since AB is a tangent to G_1 with contact point Z and AB is different from UV, we get that UV is not tangent to G_1. Hence, ΔPQR is not the second dual of Δ_0; which completes the proof. *q.e.d.*

Figure 8 shows such a configuration. Observe, that the triangles move together if one of the vertices moves along its conic. The nine vertices of the three triangles form a Pappus configuration.

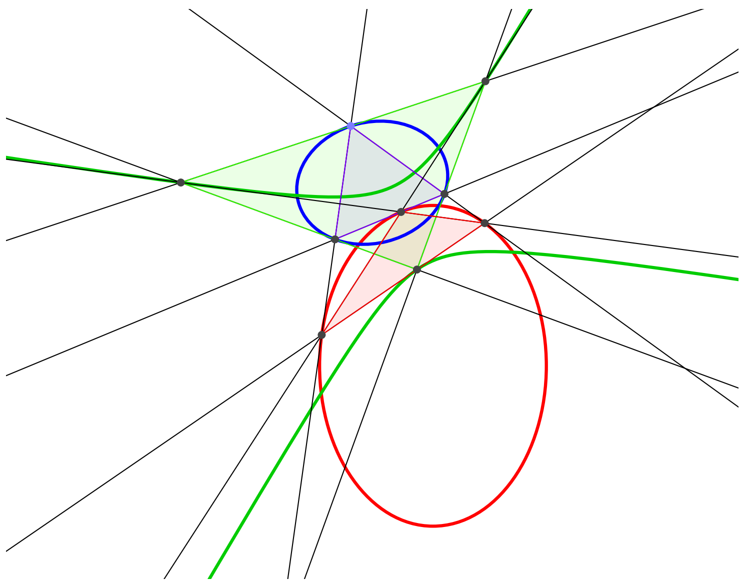

Figure 8: The red triangle is inscribed in the red ellipse and tangent to the green hyperbola. The blue triangle is inscribed in the blue ellipse and tangent to the red one. The green triangle is inscribes in the green hyperbola and tangent to the blue ellipse. Each vertex of a triangle is contact point of a side of its dual. The tangent in the point of intersection of two of the conics is tangent to the third conic. This is the limiting situation if the triangles degenerate to a line.

The situation we encountered above of a closed chain of three conjugate conics which carries a closed chain of dual Poncelet triangles is quite miraculous. Thus, we shall call such chains **miraculous chains of Poncelet triangles**. The question arises whether other miraculous chains exist. A first result is, that miraculous chains which carry Poncelet triangles must be of length 3:

PROPOSITION 3.7. *If G_0, G_1 induces a closed chain of conjugate conics of length n which carries Poncelet triangles, then $n = 3$.*

Proof. For a closed chain of conjugate conics of length n, we have by Lemma 3.4 that $D(\lambda)$ is of the form $(\lambda+\varepsilon)(\lambda^2+2\lambda a+1)$, where $a=\cos(2\pi\ell/n)$, $\varepsilon=1$ if n is odd, $\varepsilon=\pm 1$ if n is even, and where $\ell\in\{1,2,\dots,\lfloor\frac{n-1}{2}\rfloor\}$ is such that

$$\begin{cases}(\ell,n)=1 & \text{if } \varepsilon=1\\ (\ell,n/2)=1 \text{ and } \ell n/2 \text{ even} & \text{if } \varepsilon=-1.\end{cases}$$

If the chain carries Poncelet triangle, we get by the Cayley criterion for triangles that a is a solution of $3+4\varepsilon a-4a^2=0$. For $\varepsilon=1$ this implies that the possible values for a are $-1/2$ and $3/2$. Now, $a=3/2$ is impossible since $\cos(2\pi\ell/n)\neq 3/2$. So, we must have $a=-1/2$, which implies that $\ell/n=1/3$ and hence $n=3$. For $\varepsilon=-1$, the possible values for a are $1/2$ and $-3/2$. Again, $a=-3/2$ is not possible, and hence $a=1/2$ which implies $n=3$. *q.e.d.*

Before we investigate whether there are also miraculous chains of Poncelet quatrilaterals or of other Poncelet n-gons, we investigate some geometrical properties of miraculous chains of Poncelet triangles.

Given a general Pappus configuration of nine points and nine lines, one may ask whether it carries three conics as in Figure 8: Each of the three conics passes through three of the nine points, and in each of these points the conic is tangent to one of the three lines which pass through the point. However, this will in general not be the case. Brianchon's Theorem implies, that in a triangle which is tangent to a conic, the three lines joining a vertex of the triangle and the opposite contact point are concurrent:

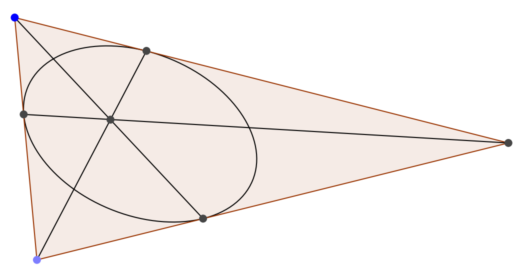

Figure 9: A consequence of Brianchon's Theorem

So, this condition must hold in each of the three triangles that are circumscribed in one of the tree conics. Surprisingly, if the condition holds in *one* of the triangles, it holds in all three triangles:

Lemma 3.8. *Let A_{ij}, $1\le i,j\le 3$ be a Pappus configuration, i.e., three points are collinear and lying on the line $\ell_{\alpha\beta}$ iff $j+\alpha i+\beta=0 \mod 3$ (see Figure 10). Furthermore, we define* *the following three quadruples H_k $(1 \le k \le 3)$ of points:*

$$\begin{aligned} H_1: &\quad \ell_{00} \wedge \ell_{01}, A_{00}, A_{10}, A_{20} \\ H_2: &\quad \ell_{01} \wedge \ell_{02}, A_{22}, A_{12}, A_{02} \\ H_3: &\quad \ell_{00} \wedge \ell_{02}, A_{01}, A_{11}, A_{21} \end{aligned}$$

Finally, we define the following three triples T_k $(1 \le k \le 3)$ of lines (see Figure 10):

$$\begin{aligned} T_1: &\quad A_{00} - A_{02}, A_{10} - A_{11}, A_{20} - A_{22} \qquad \textit{(red)} \\ T_2: &\quad A_{02} - A_{01}, A_{21} - A_{22}, A_{12} - A_{10} \qquad \textit{(green)} \\ T_3: &\quad A_{00} - A_{01}, A_{21} - A_{20}, A_{12} - A_{11} \qquad \textit{(blue)} \end{aligned}$$

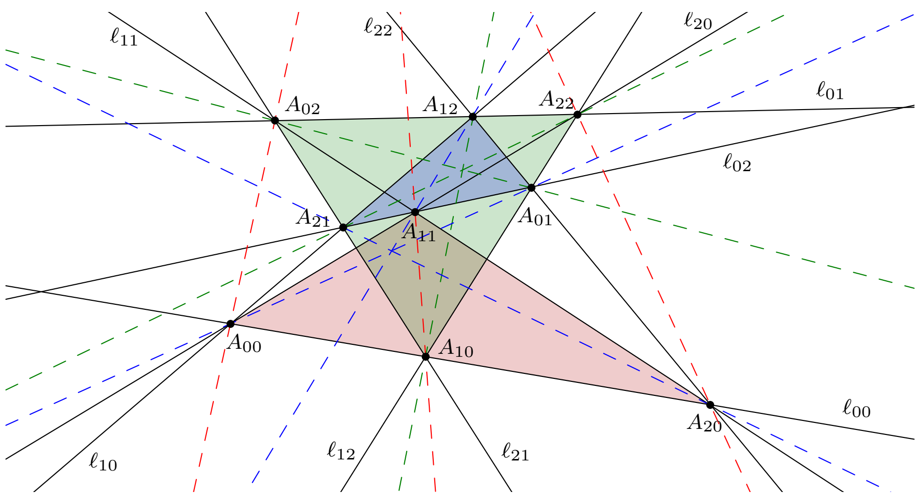

Figure 10: Pappus' Theorem. The dashed lines form the triple T_i $(1 \le i \le 3)$.

Then, for $1 \le k \le 3$, we get:

(a) T_k is concurrent iff H_k is harmonic.

(b) If one of the quadruples H_k of points is harmonic, all quadruples are harmonic.

(c) If one of the triples T_k of lines is concurrent, all triples are concurrent.

Proof. Part (a) is an immediate consequence of the theorems of Menelaos and Ceva, and (c) follows by (a) from (b). So, we just have to prove (b):

$$
\begin{array}{rcl}
 & \ell_{00} \wedge \ell_{01}, A_{00}, A_{10}, A_{20} & \boldsymbol{H_1} \textbf{ is harmonic} \\
\Rightarrow & \ell_{01}, \ell_{10}, A_{10} - A_{12}, \ell_{22} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{01} \wedge \ell_{02}, A_{01}, \ell_{02} \wedge (A_{10} - A_{12}), A_{21} & \text{are harmonic points} \\
\Rightarrow & A_{10} - (\ell_{01} \wedge \ell_{02}), \ell_{12}, A_{10} - A_{12}, \ell_{21} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{01} \wedge \ell_{02}, A_{22}, A_{12}, A_{02} & \boldsymbol{H_2} \textbf{ is harmonic} \\
\Rightarrow & \ell_{02}, \ell_{20}, A_{11} - A_{12}, \ell_{11} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{00} \wedge \ell_{02}, A_{00}, \ell_{00} \wedge (A_{11} - A_{12}), A_{20} & \text{are harmonic points} \\
\Rightarrow & A_{12} - (\ell_{00} \wedge \ell_{02}), \ell_{10}, A_{11} - A_{12}, \ell_{22} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{00} \wedge \ell_{02}, A_{21}, A_{11}, A_{01} & \boldsymbol{H_3} \textbf{ is harmonic} \\
\Rightarrow & \ell_{00}, \ell_{21}, A_{11} - A_{10}, \ell_{12} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{00} \wedge \ell_{01}, A_{02}, \ell_{01} \wedge (A_{11} - A_{10}), A_{22} & \text{are harmonic points} \\
\Rightarrow & A_{11} - (\ell_{00} \wedge \ell_{01}), \ell_{11}, A_{11} - A_{10}, \ell_{20} & \text{is a harmonic pencil} \\
\Rightarrow & \ell_{00} \wedge \ell_{01}, A_{20}, A_{10}, A_{00} & \boldsymbol{H_1} \textbf{ is harmonic}
\end{array}
$$

q.e.d.

As a consequence we get the following

COROLLARY 3.9. *Let A_{ij}, $1 \le i, j \le 3$ be a Pappus configuration, i.e., three points are collinear and lying on the line $\ell_{\alpha\beta}$ iff $j + \alpha i + \beta = 0 \mod 3$ (see Figure 10). Then, the following are equivalent:*

- *The lines $A_{00} - A_{02}, A_{10} - A_{11}, A_{20} - A_{22}$ are concurrent.*
- *The points $\ell_{00} \wedge \ell_{01}, A_{00}, A_{10}, A_{20}$ are harmonic.*
- *The configuration carries a closed Poncelet chain for triangles of length three: There are three conics C_1, C_2, C_3 such that*

  the triangle $A_{00}A_{11}A_{20}$ is inscribed in C_1 and circumscribed about C_3
  the triangle $A_{01}A_{12}A_{21}$ is inscribed in C_2 and circumscribed about C_1
  the triangle $A_{02}A_{10}A_{22}$ is inscribed in C_3 and circumscribed about C_2

We close the discussion of miraculous chains of Poncelet triangles with the following

Proposition 3.10. *Let G_0, G_1, G_2 be a closed chain of conjugate conics and $\Delta_0, \Delta_1, \Delta_2$ be three Poncelet triangles such that Δ_i has its vertices on G_i and the vertices of Δ_{i+1} are the contact points of Δ_i, where we take indices modulo 3. Then the Brianchon point of Δ_i lies on G_i.*

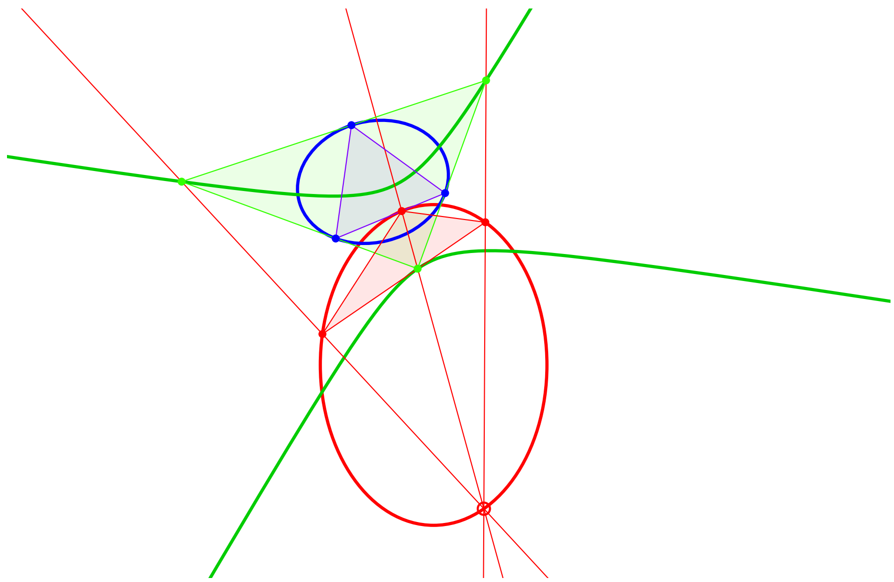

Figure 11: The Brianchon point of the red triangle lies on the red conic.

Proof. We apply Pascal's Theorem: Let P be the intersection of the triple T_2 (green in Figure 10). Consider the hexagon

$$P - A_{22} - A_{22} - A_{10} - A_{02} - A_{02}$$

Five of these points lie on the green hyperbola. The sixth point P also lies on this hyperbola if the intersections of opposite sides of the hexagon are collinear. And indeed, these intersections are $A_{21} - A_{11} - A_{01}$ (observe that two of the sides are tangents). *q.e.d.*

Let us now investigate chains of conjugate conics of length $n = 6$: From Lemma 3.4 we infer that $D(\lambda)$ is, up to a factor, one of the following polynomials:

$$\lambda^3 + 2\lambda^2 + 2\lambda + 1, \quad \lambda^3 - 2\lambda^2 + 2\lambda - 1$$

The Taylor series of the roots of these polynomials are

$$\sqrt{D(\lambda)} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^5}{8} - \frac{x^6}{16} + \frac{3x^8}{128} + \dots$$

and

$$\sqrt{D(\lambda)} = i(1 - x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{8} - \frac{x^6}{16} + \frac{3x^8}{128} - \dots)$$

In both cases, for $p = 2$, $k = 2p = 4$, the Cayley determinant is $c_3 = 0$, the coefficient of x^3. Therefore, the corresponding closed chain of conjugate conics of length $n = 6$ carries closed Poncelet quadrilaterals, and hence we have

THEOREM 3.11. *Each closed chain of conjugate conics of length $n = 6$ is a miraculous chain: It carries closed Poncelet quadrilaterals and the sixth dual of the first Poncelet quadrilateral is again the first Poncelet quadrilateral.*

REMARKS:

(a) Observe, that $(G_0^{-1}G_1)^6 = I$ is equivalent to $(G_1^{-1}G_0)^6 = I$. In particular, no matter whether we start with a Poncelet quadrilateral with vertices on G_0 which is tangential to G_1 or the other way round, we always get a closed cycle of dual Poncelet quadrilaterals.

(b) The relation $(G_0^{-1}G_1)^6 = I$ can be rewritten as $(G_0^{-1}G_1G_0^{-1}G_1)^3 = I$. Then, since we have $G_1G_0^{-1}G_1 = G_2$, we get $(G_0^{-1}G_2)^3 = I$. This means that G_0, G_2, G_4, and similarly G_1, G_3, G_5, are closed chains of conjugate conics of length 3 carrying Poncelet triangles which are entangled with the Poncelet quadrilaterals sitting the full chain $G_0, G_1, G_2, G_3, G_4, G_5$ of length 6.

Proof of Theorem 3.11. The calculations above show that each closed chain of conjugate conics of length $n = 6$ carries closed Poncelet quadrilaterals. Thus, we have only to prove that the sixth dual of the first Poncelet quadrilateral is again the first Poncelet quadrilateral.

Let G_0 and G_1 be such that $(G_0^{-1}G_1)^6 = I$. To simplify the notation let $A := G_0^{-1}G_1$, *i.e.*, $A^6 = I$, and assume $A^3 \neq I$.

Now, let x_0 and x_1 be two opposite vertices of a Poncelet quadrilateral Q on G_0 which is tangent to G_1. By definition of A we get that the image Q' of Q under A^3 is again a Poncelet quadrilateral on G_0 which is tangent to G_1, namely the sixth dual of Q. Let $y_0 := A^3x_0$ and $y_1 := A^3x_1$ be two opposite vertices of Q'. If $y_0 = x_0$ (or $y_0 = x_1$), then $y_1 = x_1$ (or $y_1 = x_0$) and we are done. Otherwise, let g_0 and g_1 be the lines joining $x_0 \,\&\, y_0$ and $x_1 \,\&\, y_1$ respectively, let h_0 and h_1 be the lines joining $x_0 \,\&\, x_1$ and $y_0 \,\&\, y_1$ respectively, and let j_0 and j_1 be the lines joining $x_0 \,\&\, y_1$ and $x_1 \,\&\, y_0$ respectively.

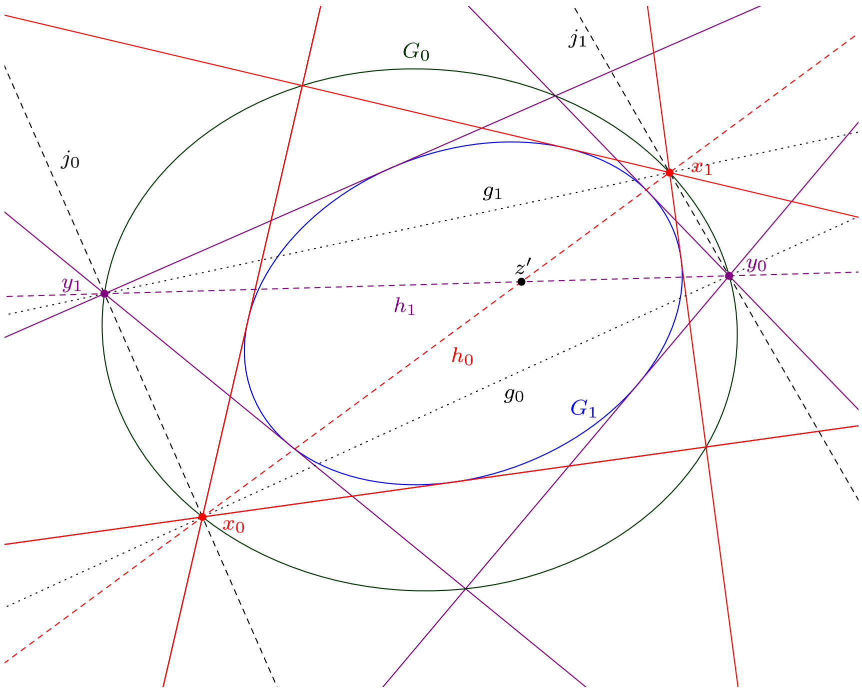

Figure 12: The two conics with the two Poncelet quadrilateral.

By definition, $A^3y_0 = x_0$, $A^3y_1 = x_1$, A^3 maps g_0 to g_0 and g_1 to g_1, A^3 maps h_0 to h_1 (and vice versa), and A^3 maps j_0 to j_1 (and vice versa). Now, let z, z', z'' be the intersecting points of g_0 & g_1, h_0 & h_1, and j_0 & j_1 respectively. Then $A^3z = z$, $A^3z' = z'$, and $A^3z' = z'$ Hence, either $A^3 = I$, which contradicts our assumption, or $x_1 = y_0$ and $x_0 = y_1$, which shows that the quadrilaterals Q and Q' are identical. *q.e.d.*

Like for triangles we can show that all miraculous chains of Poncelet quadrilaterals are of fixed length:

Proposition 3.12. *If G_0, G_1 induces a closed chain of conjugate conics of length n which carries Poncelet quadilaterals, then $n = 6$.*

Proof. By Lemma 3.4, $D(\lambda)$ is of the form $(\lambda+\varepsilon)(\lambda^2+2\lambda a+1)$, where $a = \cos(2\pi\ell/n)$, $\varepsilon = \pm 1$, and $\ell \in \{1, 2, \ldots, \lfloor\frac{n-1}{2}\rfloor\}$ satisfies

$$\begin{cases} (\ell, n) = 1 & \text{if } \varepsilon = 1 \\ (\ell, n/2) = 1 \text{ and } \ell n/2 \text{ even} & \text{if } \varepsilon = -1. \end{cases}$$

Hence, by the Cayley criterion for quadrilaterals we get that a is a solution of $8a^3 - 4\varepsilon a^2 -$

$10a + 5\varepsilon = 0$, which implies that the only possible value for a is $\varepsilon/2$. It follows that each miraculous chain of Poncelet quadrilaterals must be of length $n = 6$. *q.e.d.*

REMARK: By Lemma 3.4 together with the Cayley Criterion Theorem 3.3 one can decide if for a given n a closed chain of conjugate conics of length n which carries Poncelet k-gons exists or not. We have been looking for more such miraculous chains, but could not find any other. It is conceivable that, apart from the two cases we found in Theorem 3.6 and Theorem 3.11 respectively, no other miraculous chains exist.

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