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RESULTS ON A NAVIER-STOKES SYSTEM WITH APPLICATIONS TO ELECTRORHEOLOGICAL FLUID FLOW

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Abstract: We study a Navier-Stokes system which is motivated by models for electrorheological fluids. Its principal features are the weak monotonicity assumptions we impose on the viscosity tensor. Moreover we allow the viscosity to depend on the velocity in order to cover some of the models in electrorheological theory. We establish existence of a weak solution of the corresponding Navier-Stokes system.

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1. Introduction

1.1. A Navier-Stokes Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with Lipschitz boundary. We consider the following Navier-Stokes system for the velocity $u : \Omega \times [0, T) \to \mathbb{R}^n$ and the pressure $P : \Omega \times [0, T) \to \mathbb{R}$

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$$\frac{\partial u}{\partial t} - \operatorname{div} \sigma(x,t,u,Du) + (u \cdot \nabla)u = f - \operatorname{grad} P \text{ on } \Omega \times (0,T)\,, \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{on } \Omega \times (0,T)\,, \tag{2}$$

$$u = 0 \qquad \text{on } \partial\Omega \times (0,T)\,, \tag{3}$$

$$u(\cdot,0) = u_0 \qquad \text{on } \Omega\,. \tag{4}$$

Here, $f \in L^{p'}(0,T;V')$ for some $p \in [1 + \frac{2n}{n+2}, \infty)$, where V consists of all functions in $W_0^{1,p}(\Omega, \mathbb{R}^n)$ with vanishing divergence. Moreover $u_0 \in L^2(\Omega; \mathbb{R}^n)$, $\operatorname{div} u_0 = 0$, and σ satisfies the conditions (NS0)–(NS2) below. We allow the viscosity tensor to depend (non-linearly) on x, t, u and Du.

The problem (1)–(4) we study in this paper is motivated by the study of non-Newtonian fluid flows and, more particularly, by electrorheological fluids flows. These fluids are smart materials which are concentrated suspensions of polarizable particles in a non-conducting dielectric liquid (a typical particle size is 0.1–100μm). By applying an electric field, the viscosity can be changed by a factor up to 10^5, and the fluid can be transformed from liquid state into semi-solid state within milliseconds. The process is reversible. Examples of electrorheological fluids are alumina Al_2O_3 particles or lithiumpolymethacrylate. The phenomenon is called Winslow effect (after Willis Winslow who first investigated it in the 1940s) and is characterized by the Mason number. The Winslow effect seems to have growing applications in certain industrial sectors (see Teo [24], or Teo and Roy [25]): It is used, e.g., to construct shock absorbers in magnetically levitated trains.

In recent work (see Růžička [20], or Hoppe and Litvinov [9]) several laws for constitutive equations have been proposed for the viscosity tensor σ of such a fluid. In the model described in [9] the authors consider a situation, where σ depends on the modulus of the electric field $|E(x,t)|$, on $Du(x,t)$ but also on the angle between the velocity $u(x,t)$ of the fluid and the electric field. Moreover, for the problems studied in [9], it has been shown that the electromagnetic equations (i.e. the equations for E) are, under suitable assumptions, independent of the equations for P and u. The electric field $E(x,t)$ can therefore be considered as known and the problem may be reduced to a Navier-Stokes problem like (1)–(4). In particular, the viscosity tensor depends on x, t, u and Du.

1.2. The Main Result

Our analysis for (1)–(4) is inspired by Dolzmann, Hungerbühler and Müller [5]. By using the theory of Young measures we are able to prove an existence result

for a weak solution (u, P) to (1)–(4) under very mild assumptions on σ (see Theorem 1 below). In particular we will treat a class of problems for which the classical monotone operator methods developed by Višik [26], Minty [10], Browder [3], Brézis [2], Lions [17] and others do not apply. The reason for this is that σ does not need to satisfy the strict monotonicity condition of a typical Leray-Lions operator.

Historically, the pioneering works regarding the analysis of non-Newtonian fluid flows were realized by J. L. Lions [17] and O. Ladyzhenskaya [16]. This analysis was then extended, in particular by the works of J. Nečas and coauthors (see for instance Málek, Nečas, Rokyta, Růžička [18, 19]). See also Steinhauer [23, Chapter 3] for some more recent results and references in the field. Nevertheless, in all these studies it is always assumed that σ is strictly monotone and consequently the result we present in Theorem 1 below cannot be deduced from previous works. Moreover we allow the viscosity tensor to depend explicitly on the velocity field.

To fix some notation, let $\mathbb{M}^{m\times n}$ denote the real vector space of $m \times n$ matrices equipped with the inner product $M : N = M_{ij}N_{ij}$ (with the usual summation convention).

Now, we state our main assumptions.

(NS0) (Continuity) $\sigma : \Omega \times (0,T) \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to \mathbb{M}^{n\times n}$ is a Carathéodory function, i.e. $(x,t) \mapsto \sigma(x,t,u,F)$ is measurable for every $(u,F) \in \mathbb{R}^n \times \mathbb{M}^{n\times n}$ and $(u,F) \mapsto \sigma(x,t,u,F)$ is continuous for almost every $(x,t) \in \Omega \times (0,T)$.

(NS1) (Growth and Coercivity) There exist $c_1 \geq 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega \times (0,T))$, $\lambda_2 \in L^1(\Omega \times (0,T))$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega \times (0,T))$, $0 < \alpha < p$, such that

$$\begin{aligned} |\sigma(x,t,u,F)| &\leq \lambda_1(x,t) + c_1(|u|^{p-1} + |F|^{p-1}), \\ \sigma(x,t,u,F) : F &\geq -\lambda_2(x,t) - \lambda_3(x,t)|u|^\alpha + c_2|F|^p . \end{aligned}$$

(NS2) (Monotonicity) σ satisfies one of the following conditions:

(a) For all $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$, the map $F \mapsto \sigma(x,t,u,F)$ is a C^1-function and is monotone, i.e.

$$(\sigma(x,t,u,F) - \sigma(x,t,u,G)) : (F - G) \geq 0 ,$$

for all $(x,t) \in \Omega \times (0,T)$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{n\times n}$.

(b) There exists a function $W : \Omega \times (0,T) \times \mathbb{R}^n \times \mathbb{M}^{n\times n} \to \mathbb{R}$ such that $\sigma(x,t,u,F) = \frac{\partial W}{\partial F}(x,t,u,F)$, and $F \mapsto W(x,t,u,F)$ is convex and C^1 for all $(x,t) \in \Omega \times (0,T)$ and all $u \in \mathbb{R}^n$.

(c) σ is strictly monotone, i.e. σ is monotone and $(\sigma(x,t,u,F) - \sigma(x,t,u,G)) : (F-G) = 0$ implies $F = G$.

The Carathéodory condition (NS0) ensures that $\sigma(x,t,u(x,t), U(x,t))$ is measurable on $\Omega\times(0,T)$ for measurable functions $u : \Omega\times(0,T) \to \mathbb{R}^n$ and $U : \Omega \times (0,T) \to \mathbb{M}^{n\times n}$ (see, e.g., Zeidler [28]). (NS1) states standard growth and coercivity conditions: They are used in the construction of approximate solutions by a Galerkin method and when we pass to the limit. The strict monotonicity condition (c) in (NS2) ensures existence of weak solutions by standard methods. However, the main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (a) or (b) as it is usually assumed in previous work.

We fix the following function spaces:

$$\mathcal{V} := \{\varphi \in C_0^\infty(\Omega;\mathbb{R}^n) : \operatorname{div}\varphi = 0\}.$$

Then, V denotes the closure of $\mathcal{V}$ in the space $W^{1,p}(\Omega;\mathbb{R}^n)$. A classical result of de Rham shows, that this space is

$$V = \{\varphi \in W_0^{1,p}(\Omega;\mathbb{R}^n) : \operatorname{div}\varphi = 0\}$$

(here we use, that Ω has a Lipschitz boundary, see Galdi [8, Chapter III]). In addition, we will have to work with $W^{s,2}(\Omega;\mathbb{R}^n)$, where $s > 1 + \frac{n}{2}$. Then, we denote by

$$V_s := \text{the closure of } \mathcal{V} \text{ in the space } W^{s,2}(\Omega)$$

and

$$\begin{aligned} H_q &:= \text{the closure of } \mathcal{V} \text{ in the space } L^q(\Omega), \text{ and} \\ H &:= H_2. \end{aligned}$$

Furthermore, let $\mathcal{W}$ denote the space defined by

$$\mathcal{W} := \{v \in L^p(0,T;V) : \partial_t v \in L^{p'}(0,T;V')\},$$

where the integrals are to be understood in the sense of Bochner and the derivative means here vectorial distributional derivative (see the beginning of Section 4 and the end of Paragraph 4.3 for more information). We recall for the moment only that $\mathcal{W}$ is continuously embedded in $C^0([0,T];H)$ and we always identify $v \in \mathcal{W}$ with its representative in $C^0([0,T];H)$.

The main result we will prove is the following theorem.

Theorem 1. *Assume that σ satisfies the conditions (NS0)–(NS2) for some $p \in [1+\frac{2n}{n+2}, \infty)$. Then for every $f \in L^{p'}(0,T;V')$ and every $u_0 \in H$, the Navier-Stokes system (1)–(4) has a weak solution (u,P), with $u \in \mathcal{W}$, in the following sense: For every $v \in L^p(0,T;V)$ there holds*

$$\int_0^T \langle \partial_t u, v\rangle dt + \int_0^T \int_\Omega \sigma(x,t,u,Du) : Dv\, dx\, dt + \int_0^T \int_\Omega (u\cdot\nabla)u \cdot v\, dx\, dt = \int_0^T \langle f, v\rangle dt.$$

Remarks. (i) Let $u \in \mathcal{W}$ be a weak solution in the sense of Theorem 1. If we define the distribution $S \in (\mathcal{D}'(\Omega\times(0,T))^n$ by setting

$$S := \partial_t u - \operatorname{div}\sigma(x,t,u,Du) + (u\cdot\nabla)u - f,$$

then S satisfies $\langle S, \varphi\rangle = 0$ for all $\varphi \in \mathcal{V}$. Consequently, by a result of de Rham, $S = -\operatorname{grad} P$ for a $P \in \mathcal{D}'(\Omega\times(0,T))$ (see Galdi [8, Lemma III.1.1]). This shows that Theorem 1 ensure the existence of a classical weak solution (u,P) for problem (1)–(4).

(ii) The weak solution in Theorem 1 is more than a classical weak solution. In fact, we have $u \in C^0([0,T];H)$ and moreover the energy equality is satisfied, i.e. for all $s_2 \in [0,T]$ and all $s_1 \in [0,s_2]$ we have:

$$\frac{1}{2}\|u(\cdot,s_2)\|_H^2 + \int_{s_1}^{s_2}\int_\Omega \sigma(x,t,u,Du) : Du\, dx\, dt = \frac{1}{2}\|u(\cdot,s_1)\|_H^2 + \int_{s_1}^{s_2}\langle f,u\rangle dt.$$

1.3. Organization of the Paper

In Section 2 we indicate the choice of the Galerkin basis. In Section 3 we introduce and solve the Galerkin equations: we obtain a sequence u_m of approximating solutions for problem (1)–(4). In Section 4 we establish various properties of convergence for the sequence u_m. In particular we prove that u_m converges weakly to some u in $L^p(0,T;V)$. It is then easy to see that $-\operatorname{div}\sigma(x,t,u_m,Du_m)$ converges weakly to some χ in $L^{p'}(0,T;V')$ but the principal difficulty will be to show that $\chi = -\operatorname{div}\sigma(x,t,u,Du)$. In Section 5 we make a step in this direction by studying the Young measure associated to the sequence (u_m, Du_m), and in Section 6 we prove a div-curl inequality which is the key ingredient for Section 7. In Section 7 we pass to the limit $m\to\infty$ in the Galerkin equations and prove Theorem 1.

2. Choice of the Galerkin Base

Let $s > 1 + \frac{n}{2}$ such that for $v \in W^{s,2}(\Omega)$ we have $Dv \in W^{s-1,2}(\Omega) \subset L^\infty(\Omega)$. In particular, we have

$$V_s \subset V \subset H \cong H' \subset V' \subset V_s'$$

(where H is identified with H' by the canonical isomorphism of Hilbert spaces).

For $\zeta \in H$ we consider the linear bounded map

$$\varphi : V_s \to \mathbb{R}, \quad v \mapsto (\zeta, v)_H \,,$$

where $(\cdot,\cdot)_H$ denotes the inner product of H. By the Riesz Representation Theorem there exists a unique $K\zeta \in V_s$ such that

$$\varphi(v) = (\zeta, v)_H = (K\zeta, v)_{V_s} \qquad \text{for all } v \in V_s.$$

The map $H \to H$, $\zeta \mapsto K\zeta$, is linear, continuous, injective and (due to the compact embedding $V_s \subset H$) compact. Moreover, since

$$(\zeta, K\zeta)_H = (K\zeta, K\zeta)_{V_s} \geq 0\,,$$

the operator K is strictly positive. Hence, there exists an L^2-orthonormal base $W := \{w_1, w_2, \ldots\}$ of eigenvectors of K and strictly positive real eigenvalues λ_i with $Kw_i = \lambda_i w_i$. This means, in particular, that $w_i \in V_s$ for all i and that for all $v \in V_s$

$$\lambda_i (w_i, v)_{V_s} = (Kw_i, v)_{V_s} = (w_i, v)_H. \tag{5}$$

Notice that therefore the functions w_i are orthogonal also with respect to the inner product of V_s: In fact, for $i \neq j$, we get by choosing $v = w_j$ in (5)

$$0 = \frac{1}{\lambda_i}(w_i, w_j)_H = (w_i, w_j)_{V_s}$$

(in the finite dimensional eigenspaces, a well known theorem of linear algebra guarantees, that the vectors can be chosen orthogonal simultaneously with respect to both inner products). Notice also that, by choosing $v = w_i$ in (5),

$$1 = \|w_i\|_{L^2}^2 = (w_i, w_i)_H = \lambda_i (w_i, w_i)_{V_s} = \lambda_i \|w_i\|_{W^{s,2}}^2.$$

Thus, $\widetilde{W} := \{\tilde{w}_1, \tilde{w}_2, \ldots\}$, with $\tilde{w}_i := \sqrt{\lambda_i}\, w_i$, is an orthonormal set for $W_0^{s,2}(\Omega)$. Actually, $\widetilde{W}$ is a basis for V_s. To see this, observe that for arbitrary $v \in V_s$, the Fourier series

$$s_n(v) := \sum_{i=1}^{n} (\tilde{w}_i, v)_{W^{s,2}} \tilde{w}_i \to \tilde{v} \qquad \text{in } V_s$$

converges to some $\tilde{v}$. On the other hand, we have

$$s_n(v) = \sum_{i=1}^{n} (w_i, v)_{L^2} w_i \to v \qquad \text{in } H$$

and by the uniqueness of the limit, $\tilde{v} = v$.

We will need below the L^2-orthogonal projector $P_m : H \to H$ onto span$(w_1, w_2, \ldots, w_m)$, $m \in \mathbb{N}$. Of course, the operator norm $\|P_m\|_{\mathcal{L}(H,H)} = 1$. But notice that also $\|P_m\|_{\mathcal{L}(V_s,V_s)} = 1$ since for $u \in V_s$

$$P_m u = \sum_{i=1}^{m} (w_i, u)_H w_i = \sum_{i=1}^{m} (\tilde{w}_i, u)_{V_s} \tilde{w}_i. \tag{6}$$

3. Galerkin Approximation

We make the following assumption for approximating solutions of (1)–(4):

$$u_m(x,t) = \sum_{i=1}^{m} c_{mi}(t) w_i(x),$$

where $c_{mi} : [0,T) \to \mathbb{R}$ are supposed to be continuous bounded functions. Each u_m satisfies the side condition (2) and the boundary condition (3) by construction in the sense that $u_m \in C^0(0,T;V_s)$. We take care of the initial condition (4) by choosing the initial coefficients $c_{mi} := c_{mi}(0) = (u_0, w_i)_{L^2}$ such that

$$u_m(\cdot,0) = \sum_{i=1}^{m} c_{mi} w_i(\cdot) \to u_0 \quad \text{in } L^2(\Omega) \text{ as } m \to \infty. \tag{7}$$

We will see later on, in which sense the solution respects the initial values.

We try to determine the coefficients $c_{mi}(t)$ in such a way, that for every $m \in \mathbb{N}$ the system of ordinary differential equations

$$(\partial_t u_m, w_j)_H + \int_\Omega \sigma(x,t,u_m,Du_m) : Dw_j dx + b(u_m, u_m, w_j) = \langle f(t), w_j \rangle \tag{8}$$

(with $j \in \{1,2,\ldots,m\}$) is satisfied in the sense of distributions. In (8), $\langle \cdot, \cdot \rangle$ denotes the dual pairing of V' and V. Moreover, we used the shorthand notation

$$b(u,v,w) := \int_\Omega ((u \cdot \nabla)v) \cdot w \, dx.$$

3.1. Local Solutions for the Galerkin Equations

We fix $m \in \mathbb{N}$ for the moment. Let $0 < \varepsilon < T$ and $J = [0, \varepsilon]$. Moreover we choose $r > 0$ large enough, such that the ball $B_r(0) \subset \mathbb{R}^m$ contains the vector $(c_{m1}(0), \ldots, c_{mm}(0))$, and we set $K = \overline{B_r(0)}$. Observe that by (NS0), the function

$$\begin{aligned} F : J \times K &\to \mathbb{R}^m \\ (t, c_1, \ldots, c_m) &\mapsto \Big(\langle f(t), w_j \rangle - b(\sum_{i=1}^m c_i w_i, \sum_{i=1}^m c_i w_i, w_j) \\ &\qquad - \int_\Omega \sigma(x, t, \sum_{i=1}^m c_i w_i, \sum_{i=1}^m c_i Dw_i) : Dw_j dx \Big)_{j=1,\ldots,m} \end{aligned}$$

is a Carathéodory function. The three terms in the definition of F can easily be estimated on $J \times K$: For the first term, we have

$$|\langle f(t), w_j \rangle| \leq \|f(t)\|_{V'} \|w_j\|_V \,.$$

In the second term, we use the fact, that $Dw_j \in W^{s-1,2}(\Omega) \subset L^\infty(\Omega)$: Hence, the term $|b(\sum_{i=1}^m c_i w_i, \sum_{i=1}^m c_i w_i, w_j)|$ is bounded by a constant which depends on m and r, but not on t. For the third term, we have by the same reasoning as before and by the growth condition in (NS1)

$$|\int_\Omega \sigma(x, t, \sum_{i=1}^m c_i w_i, \sum_{i=1}^m c_i Dw_i) : Dw_j dx| \leq C \int_\Omega \lambda_1(x, t) dx + C \,,$$

where C depends on m and r but not on t. Using these estimates, we conclude that for all $j = 1, \ldots, m$ we have

$$|F_j(t, c_1, \ldots, c_m)| \leq C(r, m) M(t) \tag{9}$$

uniformly on $J \times K$, where $C(r, m)$ is a constant which depends on r and m, and where $M(t) \in L^1(J)$ does not depend on r and m. Thus, the Carathéodory existence result on ordinary differential equations (see, e.g., Kamke [13]) applied to the system

$$c_j'(t) = F_j(t, c_1(t), \ldots, c_m(t)) \,, \tag{10}$$

$$c_j(0) = c_{mj} \tag{11}$$

(for $j \in \{1, \dots, m\}$) ensures existence of a distributional, continuous solution c_j (depending on m) of (10)–(11) on a time interval $[0, \varepsilon')$, where $\varepsilon' > 0$, a priori, may depend on m. Moreover, the corresponding integral equation

$$c_j(t) = c_{mj} + \int_0^t F_j(\tau, c_1(\tau), \dots, c_m(\tau)) d\tau \tag{12}$$

holds on $[0, \varepsilon')$. Then, $u_m := \sum_{j=1}^m c_j(t) w_j$ is the desired (short time) solution of (8) with initial condition (7). We recall that Carathéodory's Theorem assures that the solutions $c_j(t)$ are absolutely continuous.

3.2. Global Solutions for the Galerkin Equations

Now, we want to show, that the local solution constructed above can be extended to the whole interval $[0, T)$ independent of m. As a word of warning we should mention, that the solution need not be unique.

The first thing we want to establish is a uniform bound on the coefficients $c_j(t)$: Since (8) is linear in w_j, it is allowed to use u_m as a test function in equation (8) in place of w_j. Observe first, that we have the identity

$$b(u_m, u_m, u_m) = \frac{1}{2} \int_\Omega u_m^i \frac{\partial}{\partial x_i} |u_m|^2 dx = -\frac{1}{2} \int_\Omega \operatorname{div} u_m |u_m|^2 dx = 0\,,$$

since $u_m(\cdot, t) \in V_s$ for all t in the existence interval (recall that all c_i are continuous functions). Therefore we get from (8) for an arbitrary time τ in the existence interval

$$\underbrace{\int_0^\tau (\partial_t u_m, u_m)_H dt}_{=:I} + \underbrace{\int_0^\tau \int_\Omega \sigma(x, t, u_m, Du_m) : Du_m dx dt}_{=:II} = \underbrace{\int_0^\tau \langle f(t), u_m \rangle}_{=:III}.$$

For the first term we have

$$I = \tfrac{1}{2} \|u_m(\cdot, \tau)\|^2_{L^2(\Omega)} - \tfrac{1}{2} \|u_m(\cdot, 0)\|^2_{L^2(\Omega)}.$$

Using the coercivity in (NS1) for the second term, we obtain

$$II \geq -\|\lambda_2\|_{L^1(\Omega \times (0,T))} - \|\lambda_3\|_{L^{(p/\alpha)'}(\Omega \times (0,T))} \|u_m\|^\alpha_{L^p(\Omega \times (0,\tau))} + c_2 \|u_m\|^p_{L^p(0,\tau;V)}.$$

For the third term, we finally get

$$III \leq \|f\|_{L^{p'}(0,T;V')} \|u_m\|_{L^p(0,\tau;V)}.$$

The combination of these three estimates gives (e.g. by using Young's inequality)

$$|(c_{mi}(\tau))_{i=1,\dots,m}|^2_{\mathbb{R}^m} = \|u_m(\cdot,\tau)\|^2_{L^2(\Omega)} \le \bar{C}\,,$$

for a constant $\bar{C}$ which is independent of τ (and of m).

Now, let

$$\Lambda := \{t \in [0,T) : \text{There exists a solution of (10)–(11) on } [0,t) \text{ in the sense specified above}\}.$$

Λ is non-empty since we proved local existence above.

Moreover Λ is an open set: To see this, let $t \in \Lambda$ and $0 < \tau_1 < \tau_2 \le t$. Then, by (12) and (9), we have

$$\begin{aligned} |c_{mj}(\tau_1) - c_{mj}(\tau_2)| &\le \int_{\tau_1}^{\tau_2} |F_j(\tau, c_{m1}(\tau), \dots, c_{mm}(\tau))| d\tau \\ &\le C(\bar{C}, m) \int_{\tau_1}^{\tau_2} |M(t)| d\tau. \end{aligned}$$

Since $M \in L^1(0,T)$, this implies that $\tau \mapsto c_{mj}(\tau)$ is uniformly continuous. Thus, we can restart to solve (8) at time t with initial data $\lim_{\tau \nearrow t} u_m(\tau)$ and hence get a solution of (10)–(11) on $[0, t+\varepsilon)$ for some $\varepsilon = \varepsilon(t) > 0$.

Finally, we prove that Λ is also closed. To see this, we consider a sequence $\tau_i \nearrow t$, $\tau_i \in \Lambda$. Let $c_{mj,i}$ denote the solution of (10)–(11) we constructed on $[0, \tau_i]$ and define

$$\tilde{c}_{mj,i}(\tau) := \begin{cases} c_{mj,i}(\tau) & \text{if } \tau \in [0, \tau_i] \\ c_{mj,i}(\tau_i) & \text{if } \tau \in (\tau_i, t). \end{cases}$$

The sequence $\{\tilde{c}_{mj,i}\}_i$ is bounded and equicontinuous on $[0,t)$, as seen above. Hence, by the Arzela-Ascoli Theorem, a subsequence (again denoted by $\tilde{c}_{mj,i}(\tau)$) converges uniformly in τ on $[0,t)$ to a continuous function $c_{mj}(\tau)$ as $i \to \infty$. Using the Lebesgue Convergence Theorem in (12) it is now easy to see that $c_{mj}(\tau)$ solves (10) on $[0,t)$. Hence $t \in \Lambda$ and thus Λ is indeed closed. And as claimed, it follows that $\Lambda = [0,T)$.

Consequently, the function $u_m(x,t) = \sum_{j=1}^m c_{mj}(t) w_j(x)$ satisfies equation (8) in the sense of $\mathcal{D}'(]0,T[)$. Moreover there exists a constant $C \ge 0$, independent of m, such that

$$\|u_m\|_{C^0([0,T];L^2(\Omega))} \le C. \tag{13}$$

4. Compactness of the Galerkin Approximation

In this section we establish various properties of convergence when $m \to \infty$ for the sequence $\{u_m\}$. Subsection 4.1 is devoted to basic convergence properties. In particular we prove that by extracting a suitable subsequence which is again denoted by u_m, we may assume that $u_m \rightharpoonup u$ in $L^p(0,T;V)$. In the Subsection 4.2 and Subsection 4.3, we prove additional convergence results for $\{u_m\}$ and a regularity result for u. This is achieved by studying the distributions associated to u_m and to u. The section ends with Subsection 4.4, where we study the convergence of $\{u_m(x,t)\}$ for fixed $t \in [0,T]$.

First we recall some facts about the vectorial distributions (see Dautray and Lions [4, p. 565–577] or Zeidler [27, p. 406–423] for a more complete introduction) and we introduce some notations.

Let X be a Banach space. The space of the vectorial distributions on $(0,T)$ to X is denoted by $\mathcal{D}'(0,T;X)$ and defined by $\mathcal{D}'(0,T;X) := \mathcal{L}(\mathcal{D}(0,T);X)$. For the functions defined on $(0,T)$ and taking values in X, we only consider in this paper the notion of integrability due to Bochner, and we denote by $L^1(0,T;X), L^p(0,T;X)$ the corresponding spaces of integrable or p-integrable functions. A fundamental result is that the space $L^1(0,T;X)$ is continuously embedded in $\mathcal{D}'(0,T;X)$. The embedding is realized by the mapping which for $f \in L^1(0,T;X)$ associates $T_f \in \mathcal{D}'(0,T;X)$, where $T_f(\varphi) = \int_0^T f(t)\varphi(t)dt$. This allows to identify T_f and f for $f \in L^1(0,T;X)$.

Every $S \in \mathcal{D}'(0,T;X)$ is differentiable in the sense of distributions, which means that the mapping $\mathcal{D}(0,T) \ni \varphi \mapsto -S(\varphi') \in X$ belongs to $\mathcal{D}'(0,T;X)$, and we denote it $\partial_t S$. Let $f \in C^1(0,T;X)$, it is easy to show that $\partial_t T_f = T_{\partial_t f}$, where $\partial_t f$ means here the classical derivative of f. Let now $f \in L^1(0,T;X)$ be such that there exists $g \in L^1(0,T;X)$ with the property that $\partial_t T_f = T_g$. According to the fact just mentioned for classically differentiable f, we will conserve the notation $\partial_t f$ for g.

4.1. Basic Convergence Properties

By testing equation (8) by u_m in place of w_j we obtain, as above in Section 3, that the sequence $\{u_m\}_m$ is bounded in

$$L^\infty(0,T;H) \cap L^p(0,T;V).$$

Therefore, by extracting a suitable subsequence which is again denoted by u_m, we may assume

$$u_m \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;H)\,, \tag{14}$$

$$u_m \rightharpoonup u \quad \text{in } L^p(0,T;V)\,. \tag{15}$$

The function $u \in L^\infty(0,T;H) \cap L^p(0,T;V)$ is a candidate to be a weak solution for the problem (1)–(4). The idea is to pass to the limit $m \to \infty$ in equation (8), but to do this, several additional properties of convergence for $\{u_m\}$ must be established. At this point, by using (15) together with the growth property for σ in (NS1) we can extract a suitable subsequence of $\{-\operatorname{div}\sigma(x,t,u_m,Du_m)\}$ such that

$$-\operatorname{div}\sigma(x,t,u_m,Du_m) \rightharpoonup \chi \quad \text{in } L^{p'}(0,T;V'), \tag{16}$$

where $\chi \in L^{p'}(0,T;V')$.

The principal difficulty will be to show that $\chi = -\operatorname{div}\sigma(x,t,u,Du)$.

4.2. Convergence in Measure

Here we shall use the Galerkin equations to get additional information on the sequence $\{\partial_t u_m\}$. The idea is then to use Aubin's Lemma in order to prove compactness of the sequence $\{u_m\}$ in an appropriate space. Technically this is achieved by the following lemma which is slightly more flexible than e.g. the version in Lions [17, Chapter 1, Section 5.2] or in Simon [22].

Lemma 2. *Let B, B_0 and B_1 be Banach spaces, B_0 and B_1 reflexive. Let $i : B_0 \to B$ be a compact linear map and $j : B \to B_1$ an injective bounded linear operator. For T finite and $1 < p_i < \infty$, $i = 0, 1$,*

$$W := \{v \mid v \in L^{p_0}(0,T;B_0),\ \partial_t(j \circ i \circ v) \in L^{p_1}(0,T;B_1)\}$$

is a Banach space under the norm $\|v\|_{L^{p_0}(0,T;B_0)} + \|\partial_t(j \circ i \circ v)\|_{L^{p_1}(0,T;B_1)}$. Then, if $V \subset W$ is bounded, the set $\{i \circ v \mid v \in V\}$ is precompact in $L^{p_0}(0,T;B)$.

A proof of Lemma 2 can be found in Hungerbühler [12].

Now, we apply Lemma 2 to the following case: $B_0 := V$, $B := H_q$ (for some q with $2 < q < p^* := \frac{np}{n-p}$ if $p < n$ and $2 < p < \infty$ if $p \geq n$) and $B_1 := V_s'$. Since we assumed that $p \geq 1 + \frac{2n}{n+2}$, we have the following chain of continuous injections:

$$B_0 \overset{i}{\hookrightarrow} B \overset{i_0}{\hookrightarrow} H \overset{\gamma}{\cong} H' \overset{i_1}{\hookrightarrow} B_1. \tag{17}$$

Here, $H \cong H'$ is the canonical isomorphism γ of the Hilbert space H and its dual. For $i : B_0 \to B$ we take simply the injection mapping, and for $j : B \to B_1$ we take the concatenation of injections and the canonical isomorphism given by (17), i.e. $j := i_1 \circ \gamma \circ i_0$. This means, that for any $u \in B_0$, the element

$j \circ i \circ u \in B_1$ is defined by the relation

$$\langle j \circ i \circ u, v \rangle = \int_\Omega uv\, dx \qquad \forall v \in V_s.$$

Then, as stated at the beginning of this section, $\{u_m\}_m$ is a bounded sequence in $L^p(0,T;B_0)$. Observe that we have $\{u_m\}_m \subset AC(0,T;V_s)$ which gives

$$\partial_t(j \circ i \circ u_m) = j \circ i \circ \partial_t u_m, \tag{18}$$

and consequently

$$\langle \partial_t(j \circ i \circ u_m), v \rangle = \int_\Omega \partial_t u_m(x,t) v(x) dx \quad \forall v \in V_s, \text{ a.e. } t \in (0,T). \tag{19}$$

On the other hand, by using (19) together with (8), and the fact that the projection operators P_m defined in (6) are selfadjoint with respect to the L^2 inner product, we obtain

$$\begin{aligned} \langle \partial_t(j \circ i \circ u_m), v \rangle &= \int_\Omega \partial_t u_m(x,t) P_m v(x) dx \\ &= -\int_\Omega \sigma(x,t,u_m,Du_m){:}D(P_m v) dx \\ &- b(u_m,u_m,P_m v) + \langle f(t), P_m v \rangle \qquad \forall v \in V_s, \text{ a.e. } t \in (0,T). \end{aligned} \tag{20}$$

Now, having established (20), we claim that indeed $\{\partial_t j \circ i \circ u_m\}_m$ is a bounded sequence in $L^{p'}(0,T;V_s')$. Namely, we have for the first term, by the growth condition in (NS1), that for $v \in V_s$

$$\begin{aligned} &|-\int_\Omega \sigma(x,t,u_m,Du_m) : DP_m v dx| \\ &\qquad \le C(\|\lambda_1(\cdot,t)\|_{L^{p'}(\Omega)} + \|u_m(\cdot,t)\|_V^{p-1}) \|P_m v\|_V . \end{aligned} \tag{21}$$

For the second term we have

$$\begin{aligned} |b(u_m,u_m,P_m v)| &= |\int_\Omega u_m^i \frac{\partial}{\partial x_i} u_m \cdot P_m v dx| \\ &= |\int_\Omega \underbrace{-\frac{\partial}{\partial x_i} u_m^i}_{=0} u_m \cdot P_m v - u_m^i u_m \cdot \frac{\partial}{\partial x_i} P_m v dx| \qquad (22) \\ &\le \underbrace{\|u_m\|^2_{L^2(\Omega)}}_{\le C \text{ for all } t \in (0,T) \text{ and all } m} \|DP_m v\|_{L^\infty(\Omega)} . \qquad (23) \end{aligned}$$

The third term contributes again

$$|\langle f, P_m v\rangle| \leq \|f(t)\|_{V'}\|P_m v\|_V. \tag{24}$$

Now, since $V_s \subset V$ we may replace $\|P_m v\|_V$ in (21) and (24) by $\|P_m v\|_{V_s}$ (up to a constant factor). Because of the remark at the end of Section 2 we have even $\|P_m v\|_{V_s} \leq \|v\|_{V_s}$. In (23) we may use, that $W^{s-1,2}(\Omega) \subset L^\infty(\Omega)$. Therefore, in (23), we can replace $\|DP_m v\|_{L^\infty(\Omega)}$ by $\|v\|_{V_s}$ (up to a constant factor).

Putting the estimates together, we obtain that

$$|\langle \partial_t(j \circ i \circ u_m), v\rangle| \leq C(t)\|v\|_{V_s}\,, \tag{25}$$

for a function $C \in L^{p'}(0,T)$. Therefore, we conclude indeed, that $\{\partial_t j \circ i \circ u_m\}_m$ is a bounded sequence in $L^{p'}(0,T;V_s')$.

Hence, from Lemma 2, we may conclude that there exists a subsequence, which we still denote by u_m, having the property that

$$u_m \to u \quad \text{in } L^p(0,T;L^q(\Omega)) \text{ for all } q < p^* \text{ and in measure on } \Omega \times (0,T). \tag{26}$$

Notice that in order to have the strong convergence simultaneously for all $q < p^*$, the usual diagonal sequence procedure applies.

4.3. A Regularity Result for u

Let $T_{j\circ i\circ u_m}, T_{j\circ i\circ u} \in \mathcal{D}'(0,T;V_s')$ be the distributions associated to $j \circ i \circ u_m$, and $j \circ i \circ u$, i.e.

$$\langle T_{j\circ i\circ u_m}(\varphi), v\rangle = \int_0^T \langle j \circ i \circ u_m, v\rangle \varphi(t)dt, \quad \forall \varphi \in \mathcal{D}(0,T), v \in V_s,$$
$$\langle T_{j\circ i\circ u}(\varphi), v\rangle = \int_0^T \langle j \circ i \circ u, v\rangle \varphi(t)dt, \quad \forall \varphi \in \mathcal{D}(0,T), v \in V_s.$$

By using the formula (19) we obtain

$$\langle \partial_t T_{j\circ i\circ u_m}(\varphi), v\rangle = -\int_\Omega \int_0^T v(x)u_m(x,t)\varphi'(t)dt\,dx,$$

which, according to (15) permits to conclude that

$$\lim_{m\to\infty} \langle \partial_t T_{j\circ i\circ u_m}(\varphi), v\rangle = \langle \partial_t T_{j\circ i\circ u}(\varphi), v\rangle. \tag{27}$$

On the other hand, according to (20) we have:

$$\langle \partial_t T_{j\circ i\circ u_m}(\varphi), v\rangle = -\int_0^T \int_\Omega \sigma(x,t,u_m,Du_m) : D(P_m v)dx\varphi(t)dt + \int_0^T \langle f(t), P_m v\rangle \varphi(t)dt - \int_0^T b(u_m,u_m,P_m v)\varphi(t)dt. \quad (28)$$

By using (16) we see that the first term in (28) converge to $-\int_0^T \langle \chi(t), v\rangle \varphi(t)dt$ when $m \to \infty$. Clearly, the second term converges to $\int_0^T \langle f(t), v\rangle \varphi(t)dt$, and we claim that

$$\lim_{m\to\infty} \int_0^T b(u_m,u_m,P_m v)\varphi(t)dt = \int_0^T b(u,u,v)\varphi(t)dt, \qquad \forall v \in V_s, \varphi \in \mathcal{D}(0,T). \quad (29)$$

This last result is easy to prove by using the formula (22) together with the property (14) and the fact that $P_m v \to v$ in V_s. It follows that, when $m \to \infty$, the term on the right in (28) tends to $\int_0^T \langle f(t) - \chi(t), v\rangle \varphi(t)dt - \int_0^T b(u,u,v)\varphi(t)dt$, and thus by (27) we obtain

$$\langle \partial_t T_{j\circ i\circ u}(\varphi), v\rangle = \int_0^T \langle f(t) - \chi(t), v\rangle \varphi(t)dt - \int_0^T b(u,u,v)\varphi(t)dt\,. \quad (30)$$

Let us now introduce the function $g : (0,T) \to V_s'$ defined by

$$\langle g(t), v\rangle = \langle f - \chi, v\rangle - b(u,u,v), \quad \forall v \in V_s, \text{ a.e. } t \in (0,T). \quad (31)$$

We shall prove that $g \in L^{p'}(0,T;V')$ which will imply that $\partial_t T_{j\circ i\circ u} = T_g$ and thus, with our convention for notation: $\partial_t j \circ i \circ u = g \in L^{p'}(0,T;V')$. We begin by introducing the function $\mathcal{B} : (0,T) \to V_s'$ by setting

$$\langle \mathcal{B}(t), v\rangle = b(u,u,v) = \int_\Omega u^i \partial_i u \cdot v dx. \quad (32)$$

Since $\chi, f \in L^{p'}(0,T;V')$ it remains to prove that we also have $\mathcal{B} \in L^{p'}(0,T;V')$. The tool needed here is the following classical interpolation lemma.

Lemma 3. *Let $1 \le r_1, r_2, s_1, s_2 \le \infty$. For any $\theta \in]0,1[$ and any ρ and α verifying*

$$\frac{1}{\rho} = \frac{1-\theta}{r_1} + \frac{\theta}{r_2}\,, \qquad \frac{1}{\alpha} = \frac{1-\theta}{s_1} + \frac{\theta}{s_2},$$

we have the following continuous injection:

$$L^{r_1}(0,T;L^{s_1}(\Omega)) \cap L^{r_2}(0,T;L^{s_2}(\Omega)) \hookrightarrow L^{\rho}(0,T;L^{\alpha}(\Omega)).$$

Moreover, for all $\varphi \in L^{r_1}(0,T;L^{s_1}(\Omega)) \cap L^{r_2}(0,T;L^{s_2}(\Omega))$ *there holds*

$$\|\varphi\|_{L^{\rho}(0,T;L^{\alpha}(\Omega))} \leq \|\varphi\|^{1-\theta}_{L^{r_1}(0,T;L^{s_1}(\Omega))}\|\varphi\|^{\theta}_{L^{r_2}(0,T;L^{s_2}(\Omega))}.$$

Let $p^* = \frac{np}{n-p}$, the functions u^i in (32) are in the space $L^p(0,T;L^{p^*}(\Omega)) \cap L^{\infty}(0,T;L^2(\Omega))$. By applying Lemma 3 with $r_1 = p, r_2 = \infty, s_1 = p^*, s_2 = 2$ and $\theta = 3-p$ we obtain $u^i \in L^{\rho}(0,T;L^{\alpha}(\Omega))$, with $\rho = \frac{p}{p-2}$ and $\alpha = \frac{2np}{5np-4n+4p-np^2-2p^2}$. Notice that we have assumed $p < 3$. In fact the case $p \geq 3$ is easier and we leave it to the reader.

Let $v \in L^{p^*}(\Omega)$, an easy application of the Hölder inequality gives

$$\|u^i v\|_{L^{p'}(\Omega)} \leq \|v\|_{L^{p^*}(\Omega)}\|u^i\|_{L^{\beta}(\Omega)} \quad \text{a.e. } t \in (0,T),$$

where $\beta = \frac{np}{np-2n+p}$.

Recall now that we have assumed $p \geq 1 + \frac{2n}{n+2}$ which implies $\beta \leq \alpha$. Thus we can replace β with α in the previous inequality. By using the Hölder inequality we obtain

$$b(u,u,v) \leq C\|v\|_{L^{p^*}(\Omega)}\|u\|_{L^{\alpha}(\Omega)}\|u\|_V \quad \text{a.e. } t \in (0,T).$$

Consequently

$$\|b(u,u,.)\|_{V'} \leq C\|u\|_{L^{\alpha}(\Omega)}\|u\|_V \quad \text{a.e. } t \in (0,T).$$

By using again the Hölder inequality we obtain

$$\left(\int_0^T \|b(u,u,.)\|^{p'}_{V'}dt\right)^{1/p'} \leq C\|u\|_{L^p(0,T;V)}\|u\|_{L^{p/(p-2)}(0,T;L^{\alpha}(\Omega))}.$$

Finally because we have chosen $\rho = p/(p-2)$ the property $\mathcal{B} \in L^{p'}(0,T;V')$ is proved. Moreover we have obtained the estimation

$$\|\mathcal{B}\|_{L^{p'}(0,T;V')} \leq C\|u\|^{p-1}_{L^p(0,T;V)}\|u\|_{L^{\infty}(0,T;L^2(\Omega))}.$$

It follows that $\partial_t(j \circ i \circ u)$ is an element of the space $L^{p'}(0,T;V')$.

Let $\mathcal{W}$ denote the space defined by

$$\mathcal{W} := \{v \in L^p(0,T;V) : \partial_t(j \circ i \circ v) \in L^{p'}(0,T;V')\}.$$

We recall that $\mathcal{W}$ is continuously embedded in $C^0([0,T];H)$ (see, e.g., Zeidler [27, p. 422] or for a particular case Dautray and Lions [4, p. 570]). In the following we always identify $v \in \mathcal{W}$ with its representative in $C^0([0,T];H)$. For all $u, v \in \mathcal{W}$ we have the generalized integration by parts formula: For all $t_1, t_2 \in [0,T]$, $t_1 \leq t_2$, there holds

$$\int_{t_1}^{t_2} \langle \partial_t j \circ i \circ u, v\rangle + \langle \partial_t j \circ i \circ v, u\rangle dt \\ = \int_\Omega u(t_2,x)v(t_2,x)dx - \int_\Omega u(t_1,x)v(t_1,x)dx. \quad (33)$$

4.4. The Limiting Time Values for u

We remark first that by the boundedness of the sequence $\partial_t(j \circ i \circ u_m)$ in $L^{p'}(0,T;V_s')$ we may extract a subsequence (not relabeled) such that

$$\partial_t(j \circ i \circ u_m) \rightharpoonup \partial_t(j \circ i \circ u) \quad \text{in } L^{p'}(0,T;V_s'). \quad (34)$$

Now let $\varphi \in C^\infty[0,T]$ and $w \in V_s$. The function $v = w\varphi$ is clearly an element of $\mathcal{W}$ and thus by (33) we get:

$$\int_0^T \langle \partial_t j \circ i \circ u, w\rangle \varphi(t)dt = -\int_0^T \langle j \circ i \circ u, w\rangle \varphi'(t)dt \\ + \int_\Omega \left(u(x,T)\varphi(T) - u(x,0)\varphi(0)\right) w(x)dx. \quad (35)$$

We also have, for any $m \in \mathbb{N}$

$$\int_0^T \langle \partial_t j \circ i \circ u_m, w\rangle \varphi(t)dt = -\int_0^T \langle j \circ i \circ u_m, w\rangle \varphi'(t)dt \\ + \int_\Omega \left(u_m(x,T)\varphi(T) - u_m(x,0)\varphi(0)\right) w(x)dx. \quad (36)$$

It follows by using (34) together with (35) and (36) that:

$$\lim_{m\to\infty} \int_\Omega \left(u_m(x,T)\varphi(T) - u_m(x,0)\varphi(0)\right) w(x)dx \\ = \int_\Omega \left(u(x,T)\varphi(T) - u(x,0)\varphi(0)\right) w(x)dx. \quad (37)$$

At this point we choose φ such that $\varphi(T) = 0$. We recall that by construction of the sequence $\{u_m\}$ we have $u_m(x,0) \to u_0(x)$ in H (see (7)) and thus by using (37) we obtain:

$$u_m(x,0) \to u_0(x) = u(x,0) \quad \text{in } H. \tag{38}$$

Recall also that the sequence $\{u_m\}$ verifies (13). Hence, we can extract a subsequence with the property that $u_m(\cdot,T) \rightharpoonup \psi$ in H, for some $\psi \in H$. By choosing now φ such that $\varphi(0) = 0$, we get from (37) $\psi = u(\cdot,T)$. Consequently we may assume that

$$u_m(\cdot,T) \rightharpoonup u(\cdot,T), \qquad \text{in } H. \tag{39}$$

In fact the property (39) can be extended to all $t \in [0,T]$, as expressed in the following lemma.

Lemma 4. *There exists a subsequence of $\{u_m\}$ (still denoted by u_m) with the property that for every $t \in [0,T]$*

$$u_m(\cdot,t) \rightharpoonup u(\cdot,t), \qquad \text{in } H. \tag{40}$$

By (26) the convergence in (40) is actually strong for almost all $t \in [0,T]$.

We prove the lemma in two steps.

In a first step we remark that the same arguments used to establish the property (39) permit, for arbitrary fixed t in $\mathbb{Q} \cap [0,T]$, to extract a subsequence verifying

$$u_m(\cdot,t) \rightharpoonup u(\cdot,t), \qquad \text{in } H.$$

The set $\mathbb{Q} \cap [0,T]$ is countable, and thus we may use a diagonal procedure in order to extract a subsequence verifying

$$u_m(\cdot,t) \rightharpoonup u(\cdot,t), \qquad \text{in } H, \quad \forall t \in \mathbb{Q} \cap [0,T]. \tag{41}$$

In a second step we use the facts that $\mathbb{Q} \cap [0,T]$ is dense in $[0,T]$, that the sequence $\{u_m\}$ has the properties (14) and (25), and that $u \in C^0([0,T];H)$ to conclude that (41) implies (40) for the same subsequence.

To see this, we fix $t \in [0,T]$ and for arbitrary $v \in H$ we set

$$J_m := \int_\Omega (u_m(x,t) - u(x,t))\, v(x) dx. \tag{42}$$

Since, by (13), the sequence $\{u_m\}$ is bounded in H, and since V_s is dense in H, it suffices to show $J_m \to 0$ for $v \in V_s$. Let $\varepsilon > 0$.

By using the density of $\mathbb{Q} \cap [0,T]$ in $[0,T]$, we can choose $t_\varepsilon \in \mathbb{Q} \cap [0,T]$ such that

$$\Big|\int_t^{t_\varepsilon} C(s)ds\Big| \leq \frac{\varepsilon}{3\|v\|_{V_s}}, \tag{43}$$

$$\|u(t_\varepsilon) - u(t)\|_{L^2(\Omega)} \leq \frac{\varepsilon}{3\|v\|_{L^2(\Omega)}}, \tag{44}$$

where the function $C \in L^{p'}(0,T)$ appearing in (43) was defined in (25). Then we have

$$\begin{aligned} J_m = &\int_\Omega (u_m(x,t_\varepsilon) - u(x,t_\varepsilon))v(x)dx + \int_\Omega (u(x,t_\varepsilon) - u(x,t))v(x)dx \\ &+ \int_\Omega (u_m(x,t) - u_m(x,t_\varepsilon)v(x))dx \\ =:&I_m + +II + III_m. \end{aligned} \tag{45}$$

By using (41) we see that I_m tends to zero as $m \to \infty$, which allows to determine $m_\varepsilon \in \mathbb{N}$ such that $|I_m|$ is bounded by $\frac{\varepsilon}{3}$ whenever $m \geq m_\varepsilon$. By using (44) we obtain that $|II|$ is bounded by $\frac{\varepsilon}{3}$. We then rewrite III_m in the following way:

$$III_m = -\int_\Omega \int_t^{t_\varepsilon} \partial_t u_m(x,s)v(x)dt\,dx = -\int_t^{t_\varepsilon} \langle \partial_t j \circ i \circ u_m, v\rangle dt.$$

Thus by using (25) together with (43) we see that $|III_m|$ is also bounded by $\frac{\varepsilon}{3}$. Consequently, we have $|J_m| \leq \varepsilon$ whenever $m \geq m_\varepsilon$. This is true for arbitrary ε and thus J_m tends to zero as $m \to \infty$.

5. The Young Measure Generated by the Galerkin Approximation

The sequence (or at least a subsequence) of the gradients Du_m generates a Young measure $\nu_{(x,t)}$, and since u_m converges in measure to u on $\Omega \times (0,T)$, the sequence (u_m, Du_m) generates the Young measure $\delta_{u(x,t)} \otimes \nu_{(x,t)}$ (see, e.g., Hungerbühler [11]). Now, we collect some facts about the Young measure ν in the following proposition:

Proposition 5. *The Young measure $\nu_{(x,t)}$ generated by the sequence $\{Du_m\}_m$ has the following properties:*

(i) $\nu_{(x,t)}$ is a probability measure on $\mathbb{M}^{n\times n}$ for almost all $(x,t) \in \Omega \times (0,T)$.

(ii) $\nu_{(x,t)}$ satisfies $Du(x,t) = \langle \nu_{(x,t)}, \mathrm{id}\rangle$ for almost every $(x,t) \in \Omega \times (0,T)$.

(iii) $\nu_{(x,t)}$ *has finite p-th moment for almost all* $(x,t) \in \Omega \times (0,T)$.

Proof. (i) The first observation is simple: To see that $\nu_{(x,t)}$ is a probability measure on $\mathbb{M}^{n\times n}$ for almost all $(x,t) \in \Omega \times (0,T)$ it suffices to recall the fact that Du_m is (in particular) a bounded sequence in $L^1(0,T;L^1(\Omega))$ and to use the fundamental theorem in Ball [1].

(ii) As we have stated at the beginning of Section 4, $\{Du_m\}_m$ is bounded in $L^p(0,T;\, L^p(\Omega))$ and we may assume that

$$Du_m \rightharpoonup Du \qquad \text{in } L^p(0,T;L^p(\Omega)).$$

On the other hand it follows that the sequence $\{Du_m\}_m$ is equiintegrable on $\Omega \times (0,T)$ and hence, by the Dunford-Pettis Theorem (see, e.g., Dunford and Schwartz [6]), the sequence is sequentially weakly precompact in $L^1(0,T;L^1(\Omega))$ which implies that

$$Du_m \rightharpoonup \langle \nu_{(x,t)}, \mathrm{id}\rangle \qquad \text{in } L^1(0,T;L^1(\Omega)).$$

Hence, we have $Du(x,t) = \langle \nu_{(x,t)}, \mathrm{id}\rangle$ for almost every $(x,t) \in \Omega \times (0,T)$.

(iii) The next thing we have to check is, that $\nu_{(x,t)}$ has finite p-th moment for almost all $(x,t) \in \Omega \times (0,T)$. To see this, we choose a cut-off function $\eta \in C_0^\infty(B_{2\alpha}(0);\mathbb{R}^m)$ with $\eta = \mathrm{id}$ on $B_\alpha(0)$ for some $\alpha > 0$. Then, the sequence

$$D(\eta \circ u_m) = (D\eta)(u_m)Du_m$$

generates a probability Young measure $\nu^\eta_{(x,t)}$ on $\Omega \times (0,T)$ with finite p-th moment, i.e.

$$\int_{\mathbb{M}^{n\times n}} |\lambda|^p d\nu^\eta_{(x,t)}(\lambda) < \infty\,,$$

for almost all $(x,t) \in \Omega \times (0,T)$. Now, for $\varphi \in C_0^\infty(\mathbb{M}^{n\times n})$ we have

$$\varphi(D(\eta \circ u_m)) \rightharpoonup \langle \nu^\eta_{(x,t)}, \varphi\rangle = \int_{\mathbb{M}^{n\times n}} \varphi(\lambda) d\nu^\eta_{(x,t)}(\lambda)$$

weakly in $L^1(0,T;L^1(\Omega))$. Rewriting the left hand side, we have also (see, e.g., Hungerbühler [11])

$$\varphi((D\eta)(u_m)Du_m) \rightharpoonup \int_{\mathbb{M}^{n\times n}} \varphi(D\eta(u(x,t))\lambda) d\nu_{(x,t)}(\lambda).$$

Hence,

$$\nu^\eta_{(x,t)} = \nu_{(x,t)} \qquad \text{if } |u(x,t)| < \alpha.$$

Since α was arbitrary, it follows that indeed $\nu_{(x,t)}$ has finite p-th moment for almost all $(x,t) \in \Omega \times (0,T)$.

6. A Navier-Stokes div-curl Inequality

In this section, we prove a Navier-Stokes version of a "div-curl Lemma" (see also Dolzmann, Hungerbühler and Müller [5, Lemma 11]), which will be the key ingredient to obtain $\chi = -\operatorname{div}\sigma(x,t,u,Du)$ and consequently to prove that u is a weak solution of (1)–(4).

6.1. The Energy Equality

A first property for χ is the following:

$$\int_{s_1}^{s_2} \langle \chi, u\rangle dt + \frac{1}{2}\|u(\cdot,s_2)\|_H^2 = \int_{s_1}^{s_2} \langle f, u\rangle dt + \frac{1}{2}\|u(\cdot,s_1)\|_H^2, \qquad \forall\ 0 \le s_1 \le s_2 \le T. \quad (46)$$

This should be obtained easily by using the results presented in the Subsection 4.4. In fact, by using the formula (33) we get

$$\int_{s_1}^{s_2} \langle \partial_t j \circ i \circ u, u\rangle dt = \frac{1}{2}\|u(\cdot,s_2)\|_H^2 - \frac{1}{2}\|u(\cdot,s_1)\|_H^2.$$

On the other hand by (31) we get:

$$\langle \partial_t j \circ i \circ u, u\rangle = \langle f - \chi, u\rangle - \underbrace{b(u,u,u)}_{=0},$$

and thus we have obtain (46).

We will prove later that $\chi = -\operatorname{div}\sigma(x,t,u,Du)$ which will imply that u is a weak solution for problem (1)–(4). Consequently the formula (46) will be interpreted as the energy equality for u.

6.2. The div-curl Inequality

Let us consider $s \in [0,T]$, and the sequence

$$I_m := \big(\sigma(x,t,u_m,Du_m) - \sigma(x,t,u,Du)\big) : \big(Du_m - Du\big).$$

We want to prove that its negative part I_m^- is equiintegrable on $\Omega \times (0,s)$. To do this, we write I_m^- in the form

$$I_m = \sigma(x,t,u_m,Du_m) : Du_m - \sigma(x,t,u_m,Du_m) : Du$$

$$- \sigma(x,t,u,Du) : Du_m + \sigma(x,t,u,Du) : Du =: II_m + III_m + IV_m + V.$$

Clearly V is equiintegrable and the sequence II_m^- is easily seen to be equiintegrable by the coercivity condition in (NS1). Then, to see equiintegrability of the sequence III_m, we take a measurable subset $S \subset \Omega \times (0,s)$ and write

$$\begin{aligned}
&\int_S |\sigma(x,t,u_m,Du_m) : Du| dxdt \leq \\
&\leq \ \Big(\int_S |\sigma(x,t,u_m,Du_m)|^{p'} dxdt\Big)^{1/p'} \Big(\int_S |Du|^p dxdt\Big)^{1/p} \\
&\leq \ C\Big(\int_S (|\lambda_1(x,t)^{p'} + |u_m|^p + |Du_m|^p) dxdt\Big)^{1/p'} \Big(\int_S |Du|^p dxdt\Big)^{1/p},
\end{aligned}$$

where we used the growth condition in (NS1) to obtain the last inequality. The first integral is uniformly bounded in m (see Section 4). The second integral is arbitrarily small if the measure of S is chosen small enough. A similar argument gives the equiintegrability of the sequence IV_m.

Having established the equiintegrability of I_m^-, it follows by the Fatou-type Lemma [5, Lemma 6], and from the fact that, by $Du_m \rightharpoonup Du$ in $L^p(0,T;L^p(\Omega))$,

$$\lim_{m\to\infty} \int_0^s \int_\Omega \sigma(x,t,u,Du) : (Du_m - Du) dxdt = 0, \tag{47}$$

that

$$\begin{aligned}
X \ &:= \ \liminf_{m\to\infty} \int_0^s \int_\Omega I_m dxdt \\
&\geq \ \int_0^s \int_\Omega \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda) : (\lambda - Du) d\nu_{(x,t)}(\lambda) dxdt. \qquad (48)
\end{aligned}$$

On the other hand, we will see next that $X \leq 0$.

Remark first that

$$\begin{aligned}
\liminf_{m\to\infty} - \int_0^s dt \int_\Omega \sigma(x,t,u_m,Du_m) : Du dx &= - \int_0^s \langle \chi, u\rangle dt \\
&= \frac{1}{2}\|u(\cdot,s)\|_H^2 - \frac{1}{2}\|u(\cdot,0)\|_H^2 - \int_0^s \langle f,u\rangle dt, \qquad (49)
\end{aligned}$$

where the last expression was obtained by using (46). In a second step, we use the Galerkin equations to obtain

$$\int_0^s dt \int_\Omega \sigma(x,t,u_m,Du_m) : Du_m dx$$

$$= \int_0^s \langle f, u_m \rangle dt - \int_0^s \underbrace{b(u_m, u_m, u_m)}_{=0} dt - \int_0^s dt \int_\Omega u_m \partial_t u_m dx$$
$$= \int_0^s \langle f, u_m \rangle dt - \frac{1}{2}\|u_m(\cdot, s)\|_H^2 + \frac{1}{2}\|u_m(\cdot, 0)\|_H^2.$$

Now by taking the limit inf in the last expression and using (38) and (40), we see that

$$\liminf_{m\to\infty} \int_0^s dt \int_\Omega \sigma(x, t, u_m, Du_m) : Du_m dx$$
$$\leq \int_0^s \langle f, u \rangle dt - \frac{1}{2}\|u(\cdot, s)\|_H^2 + \frac{1}{2}\|u(\cdot, 0)\|_H^2,$$

which in combination with (49) gives $X \leq 0$.

Finally we have proved that

$$\int_0^s \int_\Omega \int_{\mathbb{M}^{n\times n}} \sigma(x, t, u, \lambda) : \big(\lambda - Du\big) d\nu_{(x,t)}(\lambda) dx dt \leq 0,$$

and by using the Property (ii) in Proposition 5 we obtain the following lemma.

Lemma 6. (A div-curl Inequality) *The Young measure $\nu_{(x,t)}$ generated by the gradients Du_m of the Galerkin approximations u_m has the property, that for all $s \in [0, T]$:*

$$\int_0^s \int_\Omega \int_{\mathbb{M}^{n\times n}} \big(\sigma(x, t, u, \lambda) - \sigma(x, t, u, Du)\big) : \big(\lambda - Du\big) d\nu_{(x,t)}(\lambda) dx dt \leq 0. \quad (50)$$

Notice, that if we had slightly more control on the sequence $\sigma(x, t, u_m, Du_m)$ than we actually have, the lemma would follow with equality from the classical div-curl Lemma. Observe also, that we did not make use of the monotonicity condition in the proof of the lemma.

7. Passage to the Limit

Observe first that the integrand in (50) is nonnegative by monotonicity. Thus, it follows from Lemma 6 that the integrand must vanish almost everywhere with respect to the product measure $d\nu_{(x,t)} \otimes dx \otimes dt$. Hence, we have that for almost all $(x, t) \in \Omega \times (0, T)$

$$(\sigma(x, t, u, \lambda) - \sigma(x, t, u, Du)) : (\lambda - Du) = 0 \quad \text{on spt}\, \nu_{(x,t)} \qquad (51)$$

and thus

$$\operatorname{spt} \nu_{(x,t)} \subset \{\lambda \mid (\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du)) : (\lambda - Du) = 0\}. \tag{52}$$

We consider first the easiest case

Case (c). By strict monotonicity, it follows from (51) or (52) that $\nu_{(x,t)} = \delta_{Du(x,t)}$ for almost all $(x,t) \in \Omega \times (0,T)$, and hence $Du_m \to Du$ in measure on $\Omega \times (0,T)$. Consequently we have $Du_m \to Du$ a.e in $\Omega \times (0,T)$. Moreover the sequence is bounded in $L^p(\Omega \times (0,T))$ which implies (by using the Hölder inequality) that for any fixed $\alpha \in [1,p)$ the sequence is α-equiintegrable. By using next the Vitali Convergence Theorem (see for instance Schwartz [21, p. 284]) we have (up to extraction of a subsequence) $Du_m \to Du \in L^\alpha(\Omega \times (0,T))$. With the same arguments we also obtained that (up to extraction of a further subsequence) $\sigma(x,t,u_m,Du_m) \to \sigma(x,t,u,Du)$ in $L^\beta(\Omega \times (0,T))$, for all $\beta \in [1,p')$. Now, we take a test function $w \in \cup_{i\in\mathbb{N}} \operatorname{span}(w_1,\dots,w_i)$ and $\varphi \in C_0^\infty([0,T])$ in (8) and integrate over the interval $(0,T)$ and pass to the limit $m \to \infty$. The resulting equation is

$$\int_0^T \int_\Omega \partial_t u(x,t)\varphi(t)w(x)dxdt + \int_0^T \int_\Omega \sigma(x,t,u,Du) : Dw(x)\varphi(t)dxdt \\ + \int_0^T b(u,u,w\varphi)dxdt = \langle f, \varphi w\rangle,$$

for arbitrary $w \in \cup_{i\in\mathbb{N}} \operatorname{span}(w_1,\dots,w_i)$ and $\varphi \in C^\infty([0,T])$. By density of the linear span of these functions in $L^p(0,T;V)$, this proves, that u is in fact a weak solution. Hence the theorem follows in case (c).

Now, we proceed with the proof in the single cases (a) and (b) of (NS2).

Case (b). We start by showing that for almost all $(x,t) \in \Omega \times (0,T)$, the support of $\nu_{(x,t)}$ is contained in the set, where W agrees with the supporting hyper-plane $L := \{(\lambda, W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du))\}$ in $Du(x,t)$, i.e. we want to show that

$$\operatorname{spt} \nu_{(x,t)} \subset K_{(x,t)} = \{\lambda \in \mathbb{M}^{n\times n} : W(x,t.u,\lambda) \\ = W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda - Du)\}.$$

If $\lambda \in \operatorname{spt} \nu_{(x,t)}$ then by (52)

$$(1-\tau)(\sigma(x,t,u,Du) - \sigma(x,t,u,\lambda)) : (Du - \lambda) = 0 \quad \text{for all } \tau \in [0,1]. \tag{53}$$

On the other hand, by monotonicity, we have for $\tau \in [0,1]$ that

$$0 \le (1-\tau)(\sigma(x,t,u,Du + \tau(\lambda - Du)) - \sigma(x,t,u,\lambda)) : (Du - \lambda). \tag{54}$$

Subtracting (53) from (54), we get

$$0 \leq (1-\tau)(\sigma(x,t,u,Du+\tau(\lambda-Du)) - \sigma(x,t,u,Du)) : (Du-\lambda)\,, \tag{55}$$

for all $\tau \in [0,1]$. But by monotonicity, in (55) also the reverse inequality holds and we may conclude, that

$$(\sigma(x,t,u,Du+\tau(\lambda-Du)) - \sigma(x,t,u,Du)) : (\lambda-Du) = 0\,, \tag{56}$$

for all $\tau \in [0,1]$, whenever $\lambda \in \operatorname{spt} \nu_{(x,t)}$. Now, it follows from (56) that

$$\begin{aligned} W(x,t,u,\lambda) &= W(x,t,u,Du) \\ &\quad + \int_0^1 \sigma(x,t,u,Du+\tau(\lambda-Du)) : (\lambda-Du)d\tau \\ &\qquad = W(x,t,u,Du) + \sigma(x,t,u,Du) : (\lambda-Du)\,, \end{aligned}$$

as claimed.

By the convexity of W we have $W(x,t,u,\lambda) \geq W(x,t,u,Du)+\sigma(x,t,u,Du) : (\lambda-Du)$ for all $\lambda \in \mathbb{M}^{n\times n}$ and thus L is a supporting hyper-plane for all $\lambda \in K_{(x,t)}$. Since the mapping $\lambda \mapsto W(x,t,u,\lambda)$ is by assumption continuously differentiable we obtain

$$\sigma(x,t,u,\lambda) = \sigma(x,t,u,Du) \quad \text{for all } \lambda \in K_{(x,t)} \supset \operatorname{spt} \nu_{(x,t)} \tag{57}$$

and thus

$$\bar{\sigma} := \int_{\mathbb{M}^{n\times n}} \sigma(x,t,u,\lambda)\, d\nu_{(x,t)}(\lambda) = \sigma(x,t,u,Du)\,. \tag{58}$$

This shows, that $\sigma(x,t,u_m,Du_m) \rightharpoonup \sigma(x,t,u,Du)$ in $L^1(\Omega\times(0,T))$, which suffices already to pass to the limit in the Galerkin equations. However, we can even show strong convergence of the sequence $\sigma(x,t,u_m,Du_m)$:

Now consider the Carathéodory function

$$g(x,t,u,p) = |\sigma(x,t,u,p) - \bar{\sigma}(x,t)|\,.$$

The sequence $g_m(x,t) = g(x,t,u_m(x,t),Du_m(x,t))$ is equiintegrable and thus

$$g_m \rightharpoonup \bar{g} \quad \text{weakly in } L^1(\Omega\times(0,T))$$

and the weak limit $\bar{g}$ is given by

$$\bar{g}(x,t) \quad = \quad \int_{\mathbb{R}^n\times\mathbb{M}^{n\times n}} |\sigma(x,t,\eta,\lambda) - \bar{\sigma}(x,t)|\, d\delta_{u(x,t)}(\eta) \otimes d\nu_{(x,t)}(\lambda)$$

$$= \int_{\operatorname{spt}\nu_{(x,t)}} |\sigma(x,t,u(x,t),\lambda) - \bar{\sigma}(x,t)|\, d\nu_{(x,t)}(\lambda) = 0$$

by (57) and (58). Since $g_m \geq 0$ it follows that

$$g_m \to 0 \quad \text{strongly in } L^1(\Omega \times (0,T)).$$

In fact by using that g_m is bounded in $L^{p'}(\Omega \times (0,T))$ together with the Vitali Convergence Theorem, we may conclude that $\sigma(x,t,u_m,Du_m) \to \sigma(x,t,u,Du)$ in $L^\beta(\Omega \times (0,T))$ for all $\beta \in [1,p')$. This again suffices to pass to the limit in the equation and the proof of the case (b) is finished.

Case (a): We claim that in this case for almost all $(x,t) \in \Omega \times (0,T)$ the following identity holds for all $\mu \in \mathbb{M}^{n\times n}$ on the support of $\nu_{(x,t)}$:

$$\sigma(x,t,u,\lambda) : \mu = \sigma(x,t,u,Du) : \mu + (\nabla\sigma(x,t,u,Du)\mu) : (Du - \lambda), \tag{59}$$

where ∇ is the derivative with respect to the third variable of σ. Indeed, by the monotonicity of σ we have for all $\tau \in \mathbb{R}$

$$(\sigma(x,t,u,\lambda) - \sigma(x,t,u,Du+\tau\mu)) : (\lambda - Du - \tau\mu) \geq 0,$$

whence, by (51),

$$\begin{aligned} &-\sigma(x,t,u,\lambda) : (\tau\mu) \\ &\quad \geq \; -\sigma(x,t,u,Du) : (\lambda - Du) + \sigma(x,t,u,Du+\tau\mu) : (\lambda - Du - \tau\mu) \\ &\quad = \; \tau\big((\nabla\sigma(x,t,u,Du)\mu) : (\lambda - Du) - \sigma(x,t,u,Du) : \mu\big) + o(\tau). \end{aligned}$$

The claim follows from this inequality since the sign of τ is arbitrary. Since the sequence $\sigma(x,t,u_m,Du_m)$ is equiintegrable, its weak L^1-limit $\bar{\sigma}$ is given by

$$\begin{aligned} \bar{\sigma} = \int_{\operatorname{spt}\nu_{(x,t)}} \sigma(x,t,u,\lambda) d\nu_{(x,t)}(\lambda) &= \int_{\operatorname{spt}\nu_{(x,t)}} \sigma(x,t,u,Du) d\nu_{(x,t)}(\lambda) \\ &+ (\nabla\sigma(x,t,u,Du))^t \int_{\operatorname{spt}\nu_{(x,t)}} (Du - \lambda) d\nu_{(x,t)}(\lambda) = \sigma(x,t,u,Du), \end{aligned}$$

where we used (59) in this calculation. Moreover the weak convergence holds in $L^{p'}(\Omega\times(0,T))$. This finishes the proof of the case (a) and hence of the theorem.

Remarks. (i) Notice, that in case (a) we have $\sigma(x,t,u_m,Du_m) \rightharpoonup \sigma(x,t,u,Du)$, in $L^{p'}(\Omega \times (0,T))$, in case (b) we have in addition $\sigma(x,t,u_m,Du_m) \to \sigma(x,t,u,Du)$ in $L^\beta(\Omega \times (0,T))$, for all $\beta \in [1,p')$ and in case (c), we even have $Du_m \to Du$ in $L^\alpha(\Omega \times (0,T))$ for all $\alpha \in [1,p)$.

(ii) In all cases we have $\chi = -\operatorname{div}\, \sigma(x,t,u,Du)$ and thus the energy equality in the remark after Theorem 1 is obtained by using (46).

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