KANTONSSCHULE SCHAFFHAUSEN

Matura-Exam 2009

Class 4sb

Mathematics

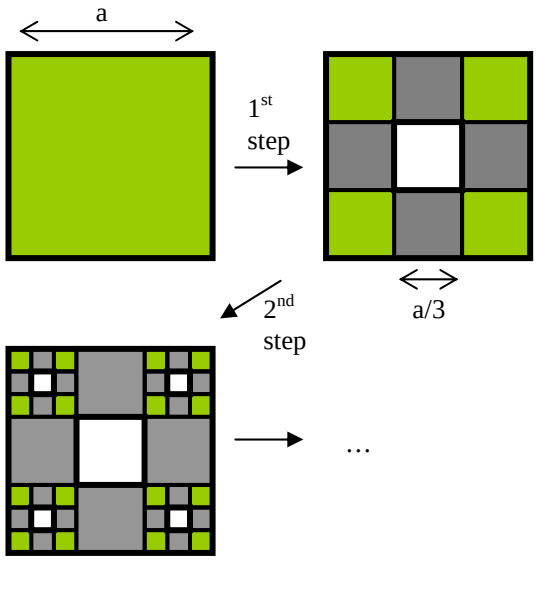
Points per question:	Question:	❶	❷	❸	❹	❺	❻	❼	❽	❾
	Points:	4/2	1/2/3	2/3	1/2/2/3/2	1/3/1/2	3/3	3/2/3/4	2/3/3	4/4

Mark scheme: max. # of points: 68 # of points to get a 6: 60 # of points to get a 4: 36

Sequences & Series

❶ We consider a square fractal which comes about as follows:

Starting from a (green) square with side length a , we remove, in the 1st step, the innermost (i.e. white) square with side length a/3 . In the 2nd step we remove, from the 4 green squares at the corners, their innermost squares with side length a/9 ; and so on.



a) Find the total area which gets removed up to and including the 6th removal step.
b) Find (without using your calculator) the total area of the fractal, i.e. the area which remains after having performed infinitely many removal steps; how many % of the original square's area is it ?

Vector Geometry

❷ We consider a right circular cone given by its apex A(-4 / 8 / 5.5) , its axis k: $\vec{r} = \begin{pmatrix} 11 \\ 3 \\ -2 \end{pmatrix} + t \cdot \begin{pmatrix} 6 \\ -2 \\ -3 \end{pmatrix}$ and a point P(-4 / 1 / 2) on the boundary of its base circle.

a) Show (without using your calculator) that A really lies on k .
b) Find the angle between k and a slant height of the cone.
c) Find the centre of the base circle.

❸ Plane F : 8x – 5y + z = 270 represents a mirror. There is another plane E which runs parallel to F and through the points A(0 / 1 / 5) and B(-1 / -2 / ?) . At point A a light ray gets directed towards the mirror F in such a way that the reflected ray meets E at point B .

a) Find the cartesian equation of E and the coordinates of point B .
b) Find the parametric equation of the reflected light ray.

Combinatorics and Probability theory

❹ In a wine producer's factory the red wine is bottled into the standard 7 dl bottles as follows: A first machine fills the empty bottles with wine, then the bottles are automatically transferred to a second machine where they get corked. As usual, these machines are not free from errors: 7 % of the bottles don't get a filling of exactly 7 dl , and only 97 out of 100 bottles get corked immaculately.

a) An empty bottle gets filled and corked. - Compute the probability (in %) that this will be done without any error.
b) Compute the number of bottles to be filled and corked so that the probability that at least one of these bottles has been treated in an erroneous way is at most equal to 60 % .
c) You buy 20 bottles of this wine. - How many of them can be expected to be in error-free condition ? And compute the probability (in %) that at least 17 out of these 20 bottles are without any error.
d) Consider the next bottle to be filled and corked. - Compute the probability (in %) that it will be treated erroneously by either the first or the second machine (but not by both !).
e) The second machine is replaced by a new model which permits to place a label on the bottles as well. As before, 97 out of 100 bottles get corked immaculately. You observe that 95 % of all the bottles get a correctly placed label, and that 91 % of all the bottles get both, i.e. a correctly placed cork and label. - Prove that the cork placing and the labelling devices are not independent of each other, i.e. prove that they certainly do influence each other.

❺ You certainly know the very popular internet search engine "Google". Due to the enormous dynamics of the internet it's unavoidable that, after having performed a Google search, some of the links being offered by Google are "dead" links, i.e. links which refer to nonexisting sites. However, Google claims that this doesn't happen too often; more precisely, Google claims that at most 8.5 % of the links being offered are "dead".
Although you are very much willing to believe that Google is right, you'd like to test the validity of their claim. To this end you randomly choose 70 of the links being offered by Google; and you note that quite many, i.e. 10 of them, are "dead".
a) State your null and alternative hypotheses in a mathematically precise way.
b) Determine C (assuming that you choose $\alpha \approx 2\ \%$).
c) Decide on the validity of H_0 !
d) Suppose that Google's claim is far from the truth, i.e. that the true rate of "dead" links is 17 % . - Compute your chances (in %) to, unfortunately, miss this fact while conducting the test described above !

Differential and Integral Calculus

❻ a) Find the equation of a polynomial function f(x) of degree 5 whose graph is symmetric about the origin, has a stationary point at P(-2 / …) and intersects the x-axis at Q(1 / …) in such a way that its tangent at Q is parallel to the line y = - 10 x .
b) The graph of $g(x) = x \cdot e^{\frac{1}{2}\left(x^3 + \ln(5)\right)}$, $0 \le x \le 2$, is rotated about the x-axis. - Compute (essentially without using your calculator) the corresponding volume of revolution.

❼ We consider, in the 2-dimensional coordinate system, the rectangle PQRS with P = (2 / 0) , Q = (2 / 16) , R = (-2 / 16) and S = (-2 / 0) . The graph of $f(x) = x^4$ divides PQRS into 2 parts, one of them lying above the graph of f(x) (with area being called A_{above}) and the other one lying below the graph of f(x) .

a) Compute A_{above} (without using your calculator).
b) Find the equation of the line t which is tangent to the graph of f(x) at Q .
c) Compute (without using your calculator) the finite area which is enclosed by the positive x-axis, the graph of f(x) and the tangent t .
d) Let's replace the function f(x) by the simpler function $g(x) = x^2$. - Compute (essentially without using your calculator) the x-coordinate of that point on the graph of g(x) , $1 \le x \le 2$, which has smallest or largest distance to Z(0 / 4) , respectively.

Differential Equations

❽ In the year 1909 the US physicist Robert Millikan conducted his famous experiments meant to determine the electric charge of the electron. To this end he observed the vertical motion of small electrically charged oil droplets whose motion was influenced by gravity and by the air's frictional force (and, sometimes, also by the force of an additional electric field which, however, will be of no concern to us). In order to evaluate his measurements he had to compute the velocity $v(t)$ of an oil droplet falling vertically towards the ground. And that's precisely what we, in this question, want to do as well.

We consider a small oil droplet (mass = $3\cdot10^{-6}$ kg) which, at time $t = 0$ and at $s(0) = 0$, begins to fall vertically towards the ground with an initial velocity of -0.2 m/s. The droplet's motion is influenced both by gravity and by the air's frictional force whose magnitude is equal to $-3\cdot10^{-7}\cdot v(t)$ (N).

a) Derive a DE for $v(t)$.

b) Let's assume that the answer of a) reads as follows: $\dot{v}(t) = -10 - 0.1\cdot v(t)$. – Calculate the solution $v(t)$ of this DE respecting the relevant IC!

c) Let's assume that the result of b) reads like this: $v(t) = \frac{72}{(6+3t)^2} - 9$ (m/s). – Derive a DE for $s(t)$ and calculate the droplet's position after the first 10 s of its fall.

❾ We consider a cotton factory with a volume of 400 m^3. The cotton fibres continuously absorb some of the air's moisture; more precisely, they absorb 3 kg of water per hour. In order to make up for these losses, there's an evaporator emitting 3.5 kg of water per hour. There is also a ventilation system replacing 100 m^3/h of air from inside the factory by fresh air from the outside (containing 5 g/m^3 of water). For the sake of simplicity we assume that the ventilation system is very efficient in the sense that the air's moisture / water is always uniformly distributed in the factory.
By $w(t)$ we denote the water's concentration (in g/m^3) in the factory air at time t. At $t = 0$, $w(0) = 15$ g/m^3.

a) Derive a DE for $w(t)$.

b) Let's assume that the solution of a) reads like this: $\dot{w}(t) = 2\,t\,w(t) + 5\,t$. – Compute the solution of this separable DE respecting the relevant IC!

Ke / 08.06.09