High-order Numerical Modeling of Highly Conductive Thin Sheets by Asymptotic Expansion

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Abstract

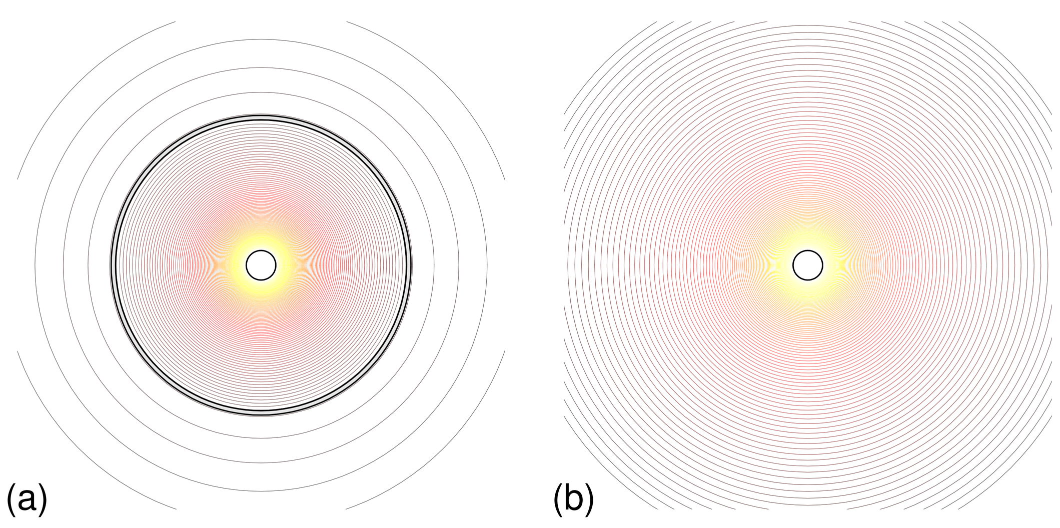
SENSITIVE measurement and control equipment is protected from disturbing electromagnetic fields by thin shielding sheets (Kost, 1994). Alternatively to discretisation of the sheets, the electromagnetic fields are modeled only in the surrounding of the layer taking them into account with transmisson conditions.

We study the shielding effect by means of the model problem of a diffusion equation with additional dissipation in the curved thin sheet. We propose asymptotic expansion models with transmission conditions for arbitrary order in the thickness ε. These models allow for highly accurate modeling of the shielding effect on meshes without cells at the scale of ε.

To numerically compute the modeling error we discretised both, the asymptotic expansion models on the limit mesh and the original problem on meshes with cells in the sheet of thickness ε. Thereby we used high-order finite elements on curved cells to diminish the effect of discretisation errors.

1. Introduction

Shielding sheets used for protection of sensitive electronic devices including integrated circuits (IC).



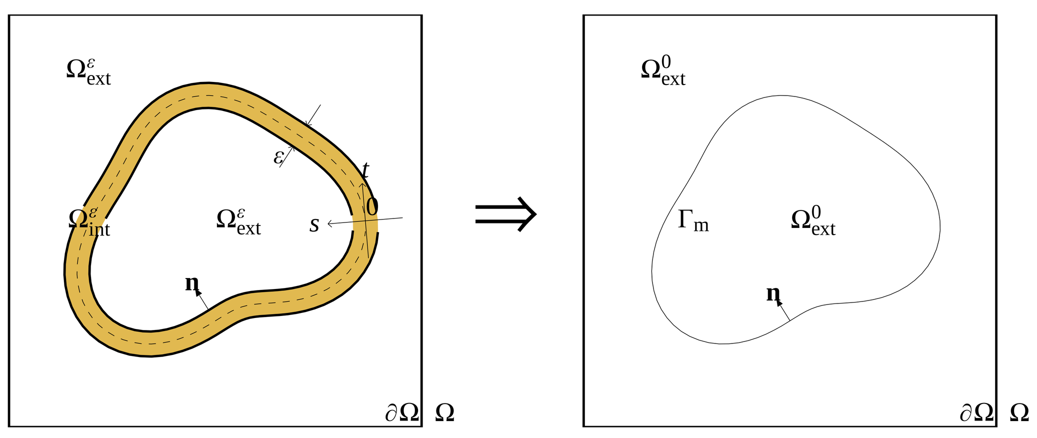
(a) (b)

Comparison of field around a live wire with and without shielding by a conducting sheet.

Issue: Geometries with sheets of small thickness are difficult or even impossible to mesh

Remedy:
1. Reduction of the sheet to its midline $\Rightarrow$ represented by edges in the mesh.
2. Enlargement of outer domain up to the midline.
3. Transmission conditions on the sheet midline to approximate the behaviour of the conductive sheet.

Reduction to sheet midline, enlargement of outer domain



Transmission conditions on sheet midline

- First order impedance boundary condition by Krähenbühl and Muller (1993), Igarashi, Kost, and Honma (1998),
- extended by a formulation with additional degrees of freedom assigned to the midline (Gyselinck & Dular, 2004).

But: Relative modeling error is in general only $O(\varepsilon)$ for simple domain enlargement (Schmidt, 2008).

2. Model Problem

Time-harmonic **Eddy-current Model** for low-frequency applications

$$\begin{cases} \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{e} + i\omega\sigma\,\mathbf{e} = -i\omega\,\mathbf{j}_0, \\ + \text{Gauge-condition}, \\ + \text{boundary conditions}. \end{cases}$$

$$\Downarrow$$

Model Problem for a particular thickness ε and conductivity c

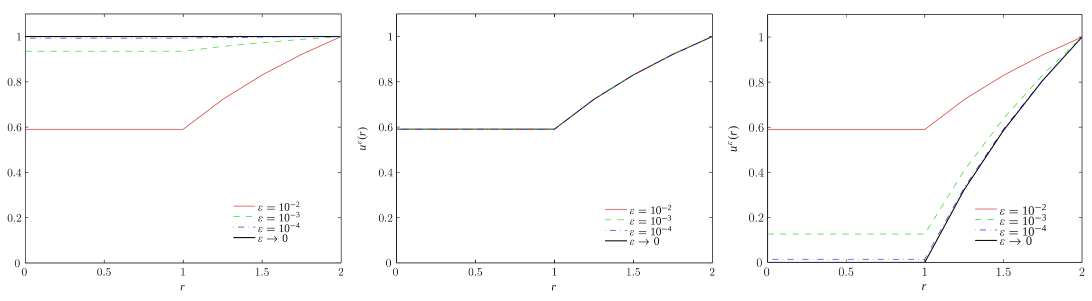
$$\begin{cases} -\Delta u + cu = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

$$\Downarrow$$

Family of problems for each thickness ε

$$\begin{cases} -\Delta u^\varepsilon_{\rm ext} = f & \text{in } \Omega^\varepsilon_{\rm ext} := \Omega\backslash\overline{\Omega}^\varepsilon_{\rm int}, \\ -\Delta u^\varepsilon_{\rm int} + \frac{c_0}{\varepsilon} u^\varepsilon_{\rm int} = 0 & \text{in } \Omega^\varepsilon_{\rm int}, \operatorname{Re} c_0 \geq 0, \\ u^\varepsilon_{\rm ext} = g & \text{on } \partial\Omega, \\ u^\varepsilon_{\rm ext} = u^\varepsilon_{\rm int} & \text{on } \Gamma^\varepsilon := \partial\Omega^\varepsilon_{\rm int}, \\ \partial_n u^\varepsilon_{\rm ext} = \partial_n u^\varepsilon_{\rm int} & \text{on } \Gamma^\varepsilon. \end{cases}$$

Choice of scaling of conductivity $c(\varepsilon) = \frac{c_0}{\varepsilon}$ because of asymptotically constant shielding



(a) $c(\varepsilon) = c_0$ (b) $c(\varepsilon) = \frac{c_0}{\varepsilon}$ (c) $c(\varepsilon) = \frac{c_0}{\varepsilon^2}$

Solutions u^ε in a section of a circular domain of radius 2 with thin sheet at $r = 1$.

Geometrical description

Consider sheet

of constant thickness ε with midline $\Gamma_{\rm m}$, given by a C^∞ parametrisation $\underline{x}_m(t)$ over interval $\hat{\Gamma}$.

Curvature denoted by $\kappa(t)$.

$\Rightarrow$ **Parametrisation of the sheet**

$$\underline{x}(t,s) = \underline{x}_m(t) + s\underline{n}, \quad s \in \left[-\varepsilon/2, -\varepsilon/2\right].$$

Goal of this work:

High order transmission conditions by means of asymptotic expansions,

see e. g. (Bendali & Lemrabet, 1996; Caloz et al., 2006).

3. Asymptotic Expansions

3.1 Expansions

Solution in stretched coordinates

Define: $S := s/\varepsilon$ and $U^\varepsilon_{\rm int}(t,S) := u^\varepsilon_{\rm int}(t, \varepsilon S)$

Asymptotic series : ansatz of power series

$$u^\varepsilon_{\rm ext}(\underline{x}) = \sum_{i=0}^\infty \varepsilon^i u^i_{\rm ext}(\underline{x}) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty)$$

$$U^\varepsilon_{\rm int}(t,S) = \sum_{i=0}^\infty \varepsilon^i U^i_{\rm int}(t,S) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty)$$

$\Rightarrow u^i_{\rm ext}(\underline{x})$ for each ε defined and so arbitrary close to midline

Move transmission condition onto midline $\Gamma_{\rm m}$

Taylor expansion around trace at $s = \pm 0$

$$u^i_{\rm ext}(t, \pm\frac{\varepsilon}{2}) = \sum_{i=0}^\infty \left(\pm\frac{1}{2}\right)^i \varepsilon^i u^i_{\rm ext}(t, \pm 0) + \underset{\varepsilon\to 0}{o}(\varepsilon^\infty)$$

Expansion of Laplace operator (in (t,S)-coordinates) in power of ε

$$\Delta = \varepsilon^{-2}\partial_S^2 + \frac{\varepsilon^{-1}\kappa(t)}{1+\varepsilon S\kappa(t)}\partial_S + \frac{1}{1+\varepsilon S\kappa(t)}\partial_t\left(\frac{1}{1+\varepsilon S\kappa(t)}\partial_t\right),$$

$$= \varepsilon^{-2}\left(\partial_S^2 + \sum_{l=1}^{L-1}\varepsilon^l \mathbf{A}_l + \varepsilon^L \mathbf{R}^L_\varepsilon\right) \quad \text{for all } L \geq 1.$$

Lemma The series $\mathbf{R}^L_\varepsilon$ converges for $L\to\infty$, if $\varepsilon < \frac{2}{\kappa(t)}$.

3.2 Hierarchical problem

Iterative solving for exterior solutions on enlarged domain with transmission conditions

The exterior functions $u^i_{\rm ext}(x)$ are given by

$$\begin{cases} -\Delta u^i_{\rm ext} = f\delta^i_0 & \text{in } \Omega^0_{\rm ext}, \\ u^i_{\rm ext} = g\delta^i_0 & \text{on } \partial\Omega, \\ [u^i_{\rm ext}](t) = \gamma^i(t) & \text{on } \Gamma_{\rm m}, \\ [\partial_s u^i_{\rm ext}](t) - c_0\{u^i_{\rm ext}\}(t) = \delta^i(t) & \text{on } \Gamma_{\rm m}. \end{cases} \tag{1}$$

with $\gamma^i(t)$, $\delta^i(t)$ functions of previous solutions $u^j_{\rm ext}$, $j < i$.

Lemma The problem (1) provides unique and stable solutions $u^i_{\rm ext} \in H^1(\Omega^0_{\rm ext})$, if $\operatorname{Re} c_0 \geq 0$.

Lemma The solutions $u^i_{\rm ext}$ are in $H^k(\Omega^0_{\rm ext})$ for any $k \in \mathbb{N}$, if $f \in H^{k-2}(\Omega^0_{\rm ext})$, $g \in H^{k-1/2}(\partial\Omega)$.

Internal expansion functions $U^i_{\rm int}(t,S)$

- are polynomials in S of order $2i$,
- follow by Sturm-Liouville problem from external functions $u^i_{\rm ext}$.

3.3 Optimal order for the modeling error

Lemma For the modeling error $r^{\varepsilon,N+1} := u^\varepsilon - \sum_{i=0}^N \varepsilon^i u^i$ holds

$$\|r^{\varepsilon,N+1}\|_{H^1(\Omega^\varepsilon_{\rm ext})} + \sqrt{\varepsilon}\|r^{\varepsilon,N+1}\|_{H^1(\Omega^\varepsilon_{\rm int})} \leq C_N \varepsilon^{N+1}.$$

Proof Problem for remainder $r^{\varepsilon,N+1}$, estimate of source terms by estimation of remainder of expansion of Laplace operator and Taylor expansion.

Lemma Same (optimal) order of ε for modeling error measured in power loss or jump of normal derivative (shielding indicators).

3.4 Concrete models

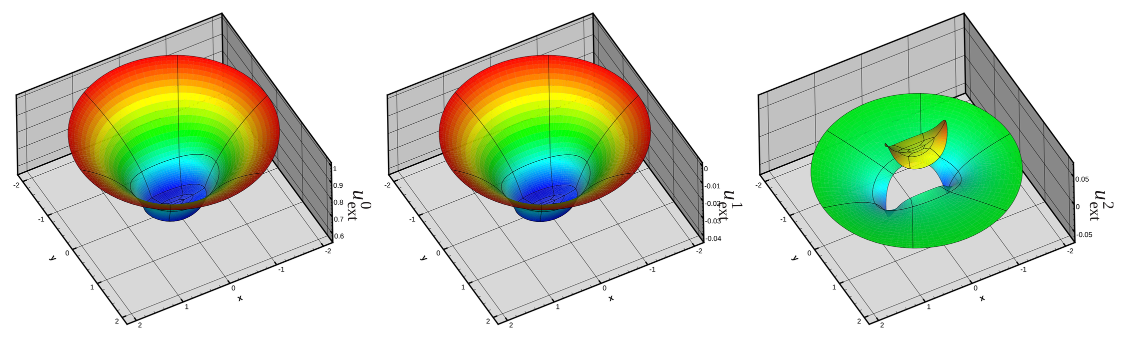
Order 0 $\gamma^0(t) = 0$, $\delta^0(t) = 0$. $\Rightarrow$ continuous over $\Gamma_{\rm m}$.

Order 1 $\gamma^1(t) = 0$, $\delta^1(t) = \frac{c_0^2}{6}u^0_{\rm ext}(t)$. $\Rightarrow$ continuous over $\Gamma_{\rm m}$.

Order 2 $\gamma^2(t) = -\frac{c_0}{24}\kappa(t)u^0_{\rm ext}(t) - \frac{c_0}{12}\left\{\partial_n u^0_{\rm ext}\right\}(t)$,

$\delta^2(t) = \frac{c_0^2}{6}u^1_{\rm ext}(t) + \frac{c_0}{24}\kappa(t)\left\{\partial_n u^0_{\rm ext}\right\}(t) + c_0\left(\frac{7}{240}c_0^2 - \frac{\partial_t^2}{12}\right)u^0_{\rm ext}(t)$.

with $\{\cdot\}(t)$ the mean of the traces from both side of the sheet.



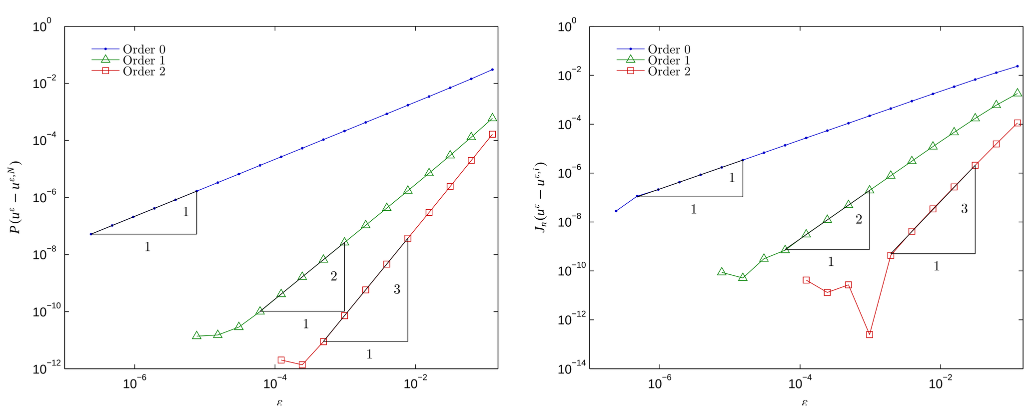
(a) Order 0 (b) Order 1 (c) Order 2

Example: Expansion functions in a circle with ellipsoidal sheet.

4. Numerical results

Implementation of exact model and asymptotic model in the Numerical C++ Library Concepts with

- use of *hp*-FE spaces,
- use of exact maps of curved edges and cells (Blending techniques), e. g. cells with circular, ellipsoidal and parallel edges.



(a) Error in power loss. (b) Error in jump in the normal derivative.

Modeling error for asymptotic expansion solutions inside a circle with ellipsoidal sheet, computed with *p*-FEM, validates theoretical estimates.

5. Collectivly computed model

Model of order 1 computed in one step for a particular ε

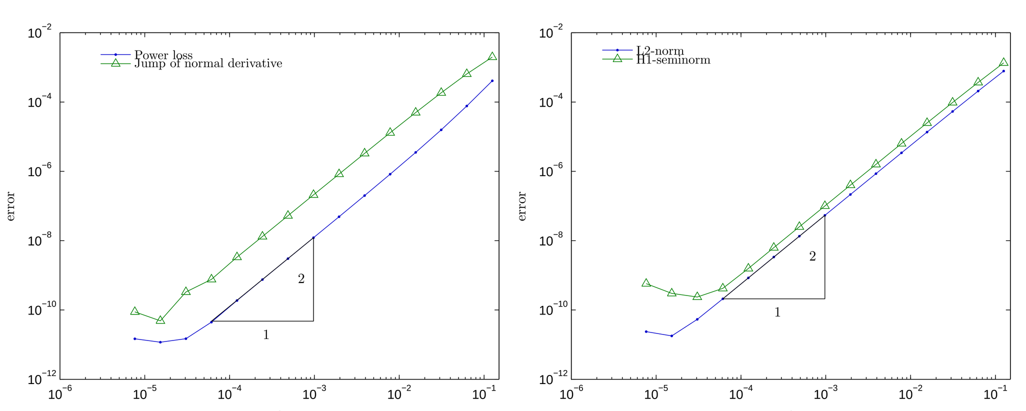
$$\begin{cases} -\Delta\tilde{u}^{\varepsilon,1}_{\rm ext} = f & \text{in } \Omega^0_{\rm ext}, \\ \tilde{u}^{\varepsilon,1}_{\rm ext} = g & \text{on } \partial\Omega, \\ [\tilde{u}^{\varepsilon,1}_{\rm ext}](t) = 0 & \text{on } \Gamma_{\rm m}, \\ \left[\partial_s\tilde{u}^{\varepsilon,1}_{\rm ext}\right](t) - c_0\left(1+\frac{c_0}{6}\right)\{\tilde{u}^{\varepsilon,1}_{\rm ext}\}(t) = 0 & \text{on } \Gamma_{\rm m}. \end{cases} \tag{2}$$

Lemma The problem (2) provides unique and stable solutions $H^1(\Omega^0_{\rm ext})$, if $\operatorname{Re} c_0 \geq 0$.

Lemma The solution $\tilde{u}^{\varepsilon,1}_{\rm ext}$ is in $H^k(\Omega^0_{\rm ext})$ for any $k \in \mathbb{N}$, if $f \in H^{k-2}(\Omega^0_{\rm ext})$, $g \in H^{k-1/2}(\partial\Omega)$.

Lemma For the modeling error it holds

$$\|u^\varepsilon - \tilde{u}^{\varepsilon,1}_{\rm ext}\|_{H^1(\Omega^\varepsilon_{\rm ext})} + \sqrt{\varepsilon}\|u^\varepsilon - \tilde{u}^{\varepsilon,1}_{\rm ext}\|_{H^1(\Omega^\varepsilon_{\rm int})} \leq C\varepsilon^2.$$



Modeling error for collectivly computed model of order 1.

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