Explicit energy conserving local time stepping for acoustic and electromagnetic wave propagation

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Outline:

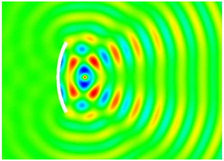
- Motivation
- Model problems
- The semi-discrete problem
- Global time stepping
- Local time stepping
- Numerical experiments
- Concluding remarks

Motivation

Modern Technology ≡ Acoustic / Electromagnetic Technology

- Electric motors and dynamos
- Antennas / Radar / Sonar
- Laser resonator
- Optical fibers
- Near field scanning optical microscopy (investigation of nano-structures)
- ...

KDD Ibaraki Satellite Communication Center



Num. Meth. for Maxwell Eq., J. Schöberl

Model Problems

Second-order wave equation

$$\begin{aligned} u_{tt} - \nabla \cdot (\, c \nabla u) &= f && \text{in } (0,T) \times \Omega, \\ u &= 0 && \text{on } (0,T) \times \partial\Omega, \\ u|_{t=0} &= u_0 && \text{in } \Omega, \\ u_t|_{t=0} &= v_0 && \text{in } \Omega. \end{aligned}$$

$\Omega \subset \mathbb{R}^2$ bounded polygon; $c(x) > 0$

Maxwell's equations in second-order form

$$\begin{aligned} \varepsilon \mathbf{u}_{tt} + \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) &= \mathbf{f} && \text{in } (0,T) \times \Omega, \\ \mathbf{n} \times \mathbf{u} &= 0 && \text{on } (0,T) \times \partial\Omega, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 && \text{in } \Omega, \\ \mathbf{u}_t|_{t=0} &= \mathbf{v}_0 && \text{in } \Omega. \end{aligned}$$

$\Omega \subset \mathbb{R}^2$ bounded polygon, a non-conducting medium; $\mu(\mathbf{x}), \varepsilon(\mathbf{x}) > 0$

System of Second-Order ODE's

The discretization in space leads to the system of ODE's

$$\mathbf{M}_\varepsilon \frac{d^2\mathbf{U}}{dt^2}(t) + \mathbf{K}_\mu \mathbf{U}(t) = \mathbf{F}(t)\,, \qquad t \in (0, T).$$

The stiffness matrix $\mathbf{K}_\mu$ and the mass matrix $\mathbf{M}_\varepsilon$ are symmetric positive (semi-)definite. For explicit time integration, $\mathbf{M}_\varepsilon$ must be (block-)diagonal $\Rightarrow$ computing $\mathbf{M}_\varepsilon^{-1}$ or $\mathbf{M}_\varepsilon^{-\frac{1}{2}}$ is cheap.

Appropriate discretizations in space

- conforming finite elements + mass-lumping techniques
- low order edge elements + mass-lumping techniques
- interior penalty discontinuous Galerkin formulation

Global Time Stepping

We consider the semi-discrete problem

$$\mathbf{M}_\varepsilon \frac{d^2}{dt^2}\mathbf{U} + \mathbf{K}_\mu \mathbf{U} = \mathbf{F}$$

and rewrite it as

$$\frac{d^2}{dt^2}\mathbf{Y} + \mathbf{A}\mathbf{Y} = \mathbf{R}$$

where $\mathbf{Y} := \mathbf{M}_\varepsilon^{\frac{1}{2}}\mathbf{U}$, $\mathbf{A} := \mathbf{M}_\varepsilon^{-\frac{1}{2}}\mathbf{K}_\mu\mathbf{M}_\varepsilon^{-\frac{1}{2}}$ and $\mathbf{R} := \mathbf{M}_\varepsilon^{-\frac{1}{2}}\mathbf{F}$.

Global Time Stepping

We consider the semi-discrete problem

$$\mathbf{M}_\varepsilon \frac{d^2}{dt^2}\mathbf{U} + \mathbf{K}_\mu \mathbf{U} = \mathbf{F}$$

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where $\mathbf{Y} := \mathbf{M}_\varepsilon^{\frac{1}{2}}\mathbf{U}$, $\mathbf{A} := \mathbf{M}_\varepsilon^{-\frac{1}{2}}\mathbf{K}_\mu \mathbf{M}_\varepsilon^{-\frac{1}{2}}$ and $\mathbf{R} := \mathbf{M}_\varepsilon^{-\frac{1}{2}}\mathbf{F}$.

The classical leap-frog scheme is given by

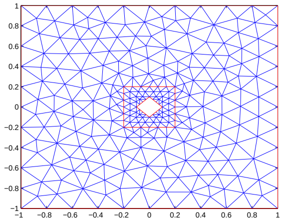
$$\mathbf{Y}^{n+1} - 2\mathbf{Y}^n + \mathbf{Y}^{n-1} = \Delta t^2(\mathbf{R}^n - \mathbf{A}\mathbf{Y}^n).$$

The scheme is stable, under the CFL condition

$$\Delta t \leq \alpha_{LF} h, \quad h = \min_{T \in \mathcal{T}_h} h_T.$$

The semi-discrete problem can be rewritten as

$$\frac{d^2}{dt^2}\mathbf{Y} + \mathbf{A}\mathbf{Y} = \mathbf{R}.$$



Let us now split $\mathbf{Y}$ and $\mathbf{R}$ in two parts

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}^{\text{coarse}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{Y}^{\text{fine}} \end{bmatrix} = (\mathbf{I} - \mathbf{P})\mathbf{Y} + \mathbf{P}\mathbf{Y}, \text{ with } \mathbf{P}^2 = \mathbf{P},$$

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}^{\text{coarse}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{R}^{\text{fine}} \end{bmatrix} = (\mathbf{I} - \mathbf{P})\mathbf{R} + \mathbf{P}\mathbf{R}, \text{ with } \mathbf{P}^2 = \mathbf{P}.$$

Then, we have

$$\frac{d^2}{dt^2}\mathbf{Y} + \mathbf{A}(\mathbf{I} - \mathbf{P})\mathbf{Y} + \mathbf{A}\mathbf{P}\mathbf{Y} = (\mathbf{I} - \mathbf{P})\mathbf{R} + \mathbf{P}\mathbf{R}.$$

Local Time Stepping

The semi-discrete problem can be rewritten as

$$\frac{d^2}{dt^2}\mathbf{Y} + \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y} + \mathbf{A}\mathbf{P}\mathbf{Y} = (\mathbf{I}-\mathbf{P})\mathbf{R} + \mathbf{P}\mathbf{R}.$$

$$\mathbf{Y}(t+\Delta t) - 2\mathbf{Y}(t) + \mathbf{Y}(t-\Delta t) = \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \frac{d^2}{dt^2}\mathbf{Y}(t+\vartheta\Delta t)d\vartheta$$

$$= \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \left\{(\mathbf{I}-\mathbf{P})\mathbf{R}(t+\vartheta\Delta t) - \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t+\vartheta\Delta t)\right\} d\vartheta$$

$$+ \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \left\{\mathbf{P}\mathbf{R}(t+\vartheta\Delta t) - \mathbf{A}\mathbf{P}\mathbf{Y}(t+\vartheta\Delta t)\right\} d\vartheta$$

$$\approx \Delta t^2 \left\{(\mathbf{I}-\mathbf{P})\mathbf{R}(t) - \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t)\right\}$$

$$+ \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \left\{\mathbf{P}\mathbf{R}(t+\vartheta\Delta t) - \mathbf{A}\mathbf{P}\mathbf{Y}(t+\vartheta\Delta t)\right\} d\vartheta$$

Local Time Stepping

$$\mathbf{Y}(t+\Delta t) - 2\mathbf{Y}(t) + \mathbf{Y}(t-\Delta t) \approx \Delta t^2 \left\{ (\mathbf{I}-\mathbf{P})\mathbf{R}(t) - \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t) \right\}$$
$$+ \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \left\{ \mathbf{P}\mathbf{R}(t+\vartheta\Delta t) - \mathbf{A}\mathbf{P}\widetilde{\mathbf{Y}}(\vartheta\Delta t) \right\} d\vartheta$$

Where $\widetilde{\mathbf{Y}}$ is the solution of

$$\left| \begin{aligned} &\widetilde{\mathbf{Y}}(0) = \mathbf{Y}(t) \\ &\widetilde{\mathbf{Y}}'(0) = \mathcal{V} \\ &\frac{d^2}{d\tau^2}\widetilde{\mathbf{Y}}(\tau) = (\mathbf{I}-\mathbf{P})\mathbf{R}(t) - \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t) \\ &\qquad\qquad + \mathbf{P}\mathbf{R}(t+\tau) - \mathbf{A}\mathbf{P}\widetilde{\mathbf{Y}}(\tau) \end{aligned} \right.$$

$$\widetilde{\mathbf{Y}}(\Delta t) - 2\widetilde{\mathbf{Y}}(0) + \widetilde{\mathbf{Y}}(-\Delta t) = \Delta t^2 \left\{ (\mathbf{I}-\mathbf{P})\mathbf{R}(t) - \mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t) \right\}$$
$$+ \Delta t^2 \int_{-1}^{1} (1-|\vartheta|) \left\{ \mathbf{P}\mathbf{R}(t+\vartheta\Delta t) - \mathbf{A}\mathbf{P}\widetilde{\mathbf{Y}}(\vartheta\Delta t) \right\} d\vartheta$$

Local Time Stepping

$$\mathbf{Y}(t+\Delta t)+\mathbf{Y}(t-\Delta t)\approx\widetilde{\mathbf{Y}}(\Delta t)+\widetilde{\mathbf{Y}}(-\Delta t)$$

$\widetilde{\mathbf{Y}}(\Delta t)+\widetilde{\mathbf{Y}}(-\Delta t)$ does not depend on the value of $\mathcal{V}$, which can be chosen arbitrarily.

$$\mathbf{Q}(\tau)=\widetilde{\mathbf{Y}}(\tau)+\widetilde{\mathbf{Y}}(-\tau)$$

$\mathbf{Q}$ is the solution of

$$\left|\begin{array}{l}\mathbf{Q}(0)=2\mathbf{Y}(t)\\ \mathbf{Q}'(0)=0\\ \dfrac{d^2}{d\tau^2}\mathbf{Q}(\tau)=2\left\{(\mathbf{I}-\mathbf{P})\mathbf{R}(t)-\mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}(t)\right\}\\ \qquad\qquad +\mathbf{PR}(t+\tau)+\mathbf{PR}(t-\tau)-\mathbf{APQ}(\tau)\end{array}\right.$$

$$\mathbf{Y}(t+\Delta t)+\mathbf{Y}(t-\Delta t)\approx\mathbf{Q}(\Delta t)$$

Local Time Stepping

$$\mathbf{Y}(t+\Delta t)+\mathbf{Y}(t-\Delta t)\approx\widetilde{\mathbf{Y}}(\Delta t)+\widetilde{\mathbf{Y}}(-\Delta t)$$

$\widetilde{\mathbf{Y}}(\Delta t)+\widetilde{\mathbf{Y}}(-\Delta t)$ does not depend on the value of $\mathcal{V}$, which can be chosen arbitrarily.

$$\mathbf{Q}(\tau)=\widetilde{\mathbf{Y}}(\tau)+\widetilde{\mathbf{Y}}(-\tau)$$

$\mathbf{Q}$ is the solution of

$$\left|\begin{array}{l} \mathbf{Q}(0)=2\mathbf{Y}^n \\ \mathbf{Q}'(0)=0 \\ \dfrac{d^2}{d\tau^2}\mathbf{Q}(\tau)=2\left\{(\mathbf{I}-\mathbf{P})\mathbf{R}^n-\mathbf{A}(\mathbf{I}-\mathbf{P})\mathbf{Y}^n\right\} \\ \qquad\qquad +\mathbf{P}\mathbf{R}(t_n+\tau)+\mathbf{P}\mathbf{R}(t_n-\tau)-\mathbf{A}\mathbf{P}\mathbf{Q}(\tau) \end{array}\right.$$

$$\mathbf{Y}^{n+1}+\mathbf{Y}^{n-1}=\mathbf{Q}(\Delta t)$$

Local Time Stepping / Algorithm

We solve the problem (Q) from $\tau = 0$ to $\tau = \Delta t$, using a Leap-Frog scheme with $\Delta\tau = \Delta t / p$.

$$\mathbf{w} = (\mathbf{I} - \mathbf{P})\mathbf{R}^n - \mathbf{A}(\mathbf{I} - \mathbf{P})\mathbf{Y}^n$$

$$\mathbf{Q}^0 = 2\mathbf{Y}^n$$

$$\mathbf{Q}^{\frac{1}{p}} = \mathbf{Q}^0 + \frac{1}{2}\left(\frac{\Delta t}{p}\right)^2 (2\mathbf{w} + 2\mathbf{P}\mathbf{R}^n - \mathbf{A}\mathbf{P}\mathbf{Q}_0)$$

$$\mathbf{Q}^{\frac{i+1}{p}} = 2\mathbf{Q}^{\frac{i}{p}} - \mathbf{Q}^{\frac{i-1}{p}} + \left(\frac{\Delta t}{p}\right)^2 \left(2\mathbf{w} + \mathbf{P}(\mathbf{R}^{n,m} + \mathbf{R}^{n,-m}) - \mathbf{A}\mathbf{P}\mathbf{Q}^{\frac{i}{p}}\right)$$

$$i = 1 \ldots p - 1$$

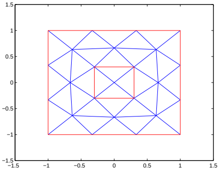
$$\mathbf{Y}^{n+1} + \mathbf{Y}^{n-1} = \mathbf{Q}(\Delta t) \quad \Longrightarrow \quad \mathbf{Y}^{n+1} = -\mathbf{Y}^{n-1} + \mathbf{Q}^1$$

This algorithm requires only one multiplication by $\mathbf{A}(\mathbf{I} - \mathbf{P})$ and p multiplications by $\mathbf{AP}$ per time-step Δt.

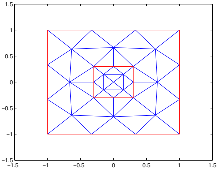
The local time-stepping scheme is second-order accurate in time.

Numerical Experiments / Example 1

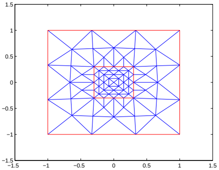
- Second-order vector wave equation
- Computational domain: $(-1, 1)^2 \times (0, 0.5)$ with local refinement ($p = 2$, $p = 4$)



global refinement



local refinement $p = 2$



local refinement $p = 4$

Numerical Experiments / Example 1

- Second-order vector wave equation
- Computational domain: $(-1,1)^2 \times (0, 0.5)$ with local refinement ($p = 2$, $p = 4$)
- Exact solution: ($\varepsilon, \mu \equiv 1$)

$$\mathbf{u}(x,y,t) = \frac{t^2}{2} \begin{bmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{bmatrix}$$

- Right-hand side:

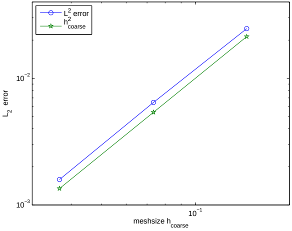
$$\mathbf{f}(x,y,t) = (1+\pi^2 t^2) \begin{bmatrix} \cos(\pi x)\sin(\pi y) \\ -\sin(\pi x)\cos(\pi y) \end{bmatrix}$$

- Homogeneous boundary condition
- Homogeneous initial conditions
- Space DG discretization with $\mathcal{P}^1$ elements

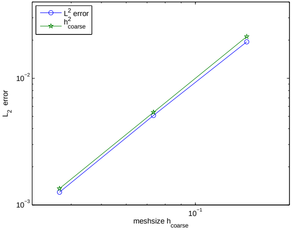
Numerical Experiments / Example 1

- Errors with respect to the L^2-norm time $T = 0.5$ for the DG approximation

level	global ref.	local ref. $p = 2$	local ref. $p = 4$
1	2.6908e-02	2.4638e-02	1.9378e-02
2	6.9554e-03	6.4469e-03	5.0874e-03
3	1.7116e-03	1.5872e-03	1.2615e-03



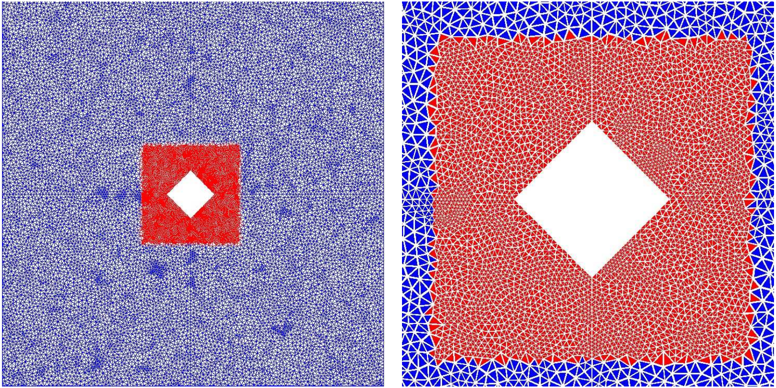
local refinement $p = 2$



local refinement $p = 4$

Numerical Experiments / Example 2

- Second-order wave equation
- Computational domain: $\Omega \times (0, 4.5)$; Ω a square of size 4×4 with a square hole of diagonal 0.25 at its center



local refinement $p = 2$

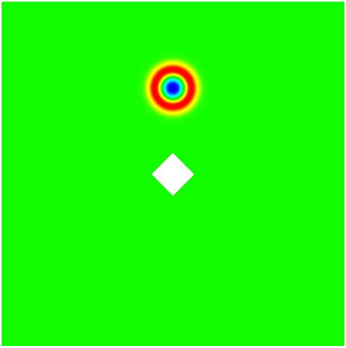
Numerical Experiments / Example 2

- Second-order wave equation
- Computational domain: $\Omega \times (0, 4.5)$; Ω a square of size 4×4 with a square hole of diagonal 0.25 at its center
- Homogeneous boundary condition
- Homogeneous source data
- The wave is excited through the inhomogeneous initial condition:

$$u|_{t=0} = e^{-\frac{\|\mathbf{x}-\mathbf{x}_0\|}{r^2}} \; (\mathbf{x}_0 = (0, 1), r = 0.1), \quad u_t|_{t=0} = 0.$$

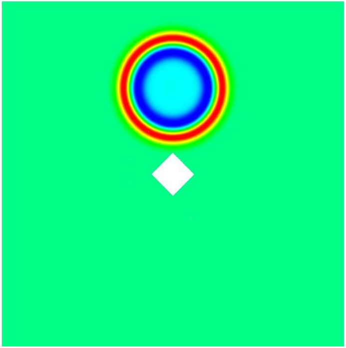
- Space DG discretization with $\mathcal{P}^3$ elements

Solution



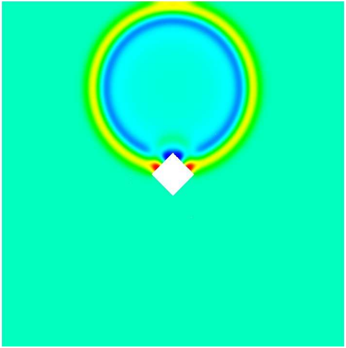
$t = 0.18$

Solution



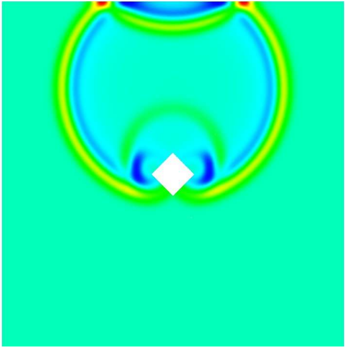
$t = 0.45$

Solution



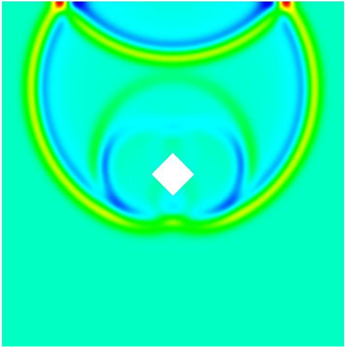
$t = 0.90$

Solution



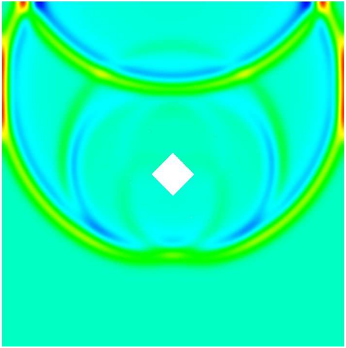
$t = 1.35$

Solution



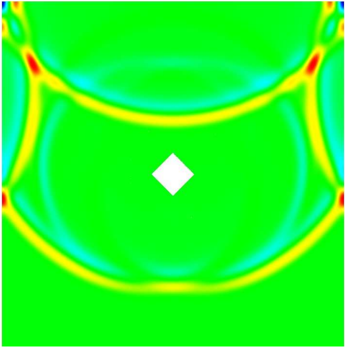
$t = 1.8$

Solution



$t = 2.25$

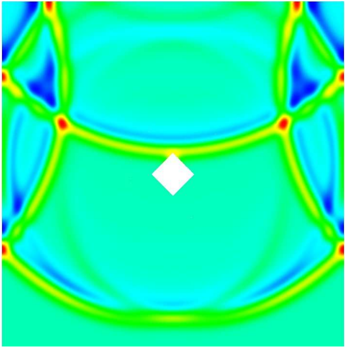
Solution



$t = 2.7$

Numerical Experiments / Example 2

Solution



$t = 3.15$

Concluding Remarks

- Two model problems
  - second-order scalar wave equation
  - Maxwell's equations in second-order form
- $\mathbf{M}_\varepsilon$ must be (block-)diagonal $\Rightarrow$ explicit time integration
- Explicit local time stepping method
  - second-order accurate
  - conservation of a discrete energy
  - generalized to arbitrary order and conducting medium

- J. Diaz and M.J. Grote, *Energy Conserving Explicit Local Time-Stepping for Second-Order Wave Equations*, Preprint 2007-02, Dept. Mathematics, University of Basel.
  see `www.math.unibas.ch/preprints`
- M.J. Grote and T. Mitkova, *Explicit Local Time-Stepping Method for Maxwell's Equations, in preparation.*