Ordering-based approaches for improving solver efficiency in reservoir simulation

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1. Motivation

Our goal is to improve the efficiency of linear and nonlinear solvers in petroleum reservoir simulators by exploiting physical characteristics intrinsic to the problem:

- Flow is driven by pressure gradients, so an upstream-to-downstream ordering is possible;
- Upwind discretization allows partial resolution of the nonlinear system on a cell-by-cell basis.

2. Flow in porous media

The flow of n immiscible phases in heterogeneous porous media is modelled by n nonlinear conservation laws defined over $\mathbf{x} \in \mathbb{R}^k$, $1 \le k \le 3$:

$$\frac{\partial(\phi\rho^j S^j)}{\partial t} + \nabla\cdot(\rho^j \mathbf{v}^j) = \rho^j q^j, \qquad j = 1,\dots,n, \tag{1}$$

with phase velocities $\mathbf{v}^j$ given by Darcy's law

$$\mathbf{v}^j = -K\lambda^j(S^1,\dots,S^n)\left[\nabla p - \rho^j \mathbf{g}\right]. \tag{2}$$

We close the system with initial and boundary conditions, as well as the *saturation constraint* $\sum_j S^j = 1$. The primary variables are p (pressure) and S^j for $j = 1,\dots,n-1$ (saturation). The equations are discretized using:

- Finite volumes with conservative numerical fluxes,
- Phase-based upwinding for $\lambda^j(S^1,\dots,S^n)$ (Fig. 3),
- Implicit time stepping (Backward Euler) for both S^j and p.

The standard nonlinear solver is Newton's method, which is locally quadratically convergent but can diverge for bad initial guesses.

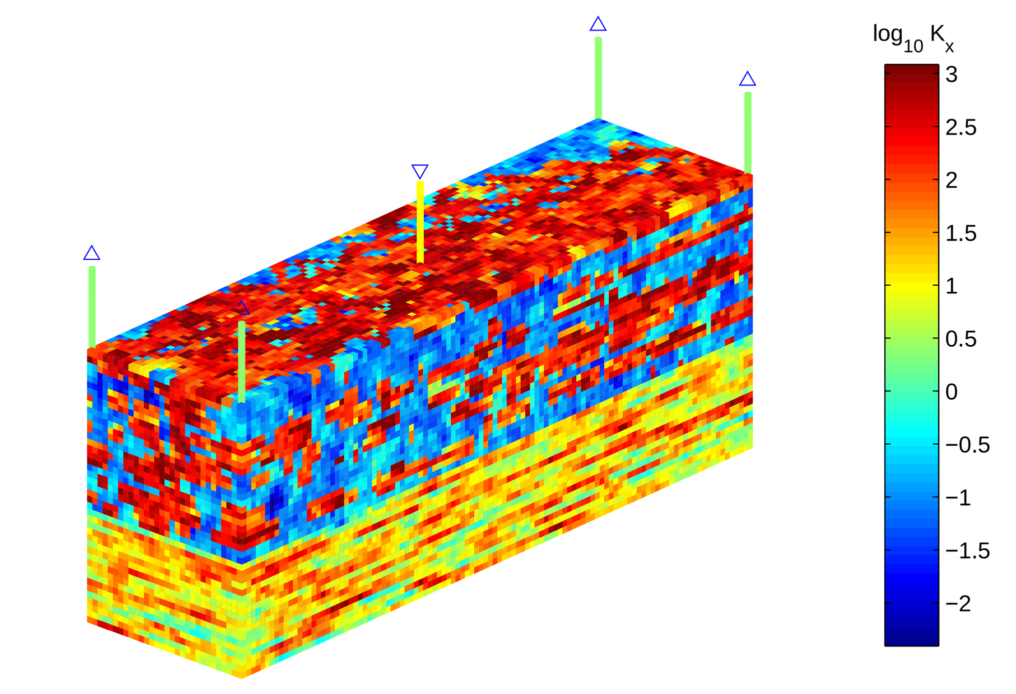


Figure 1: *A heterogeneous oil reservoir.*

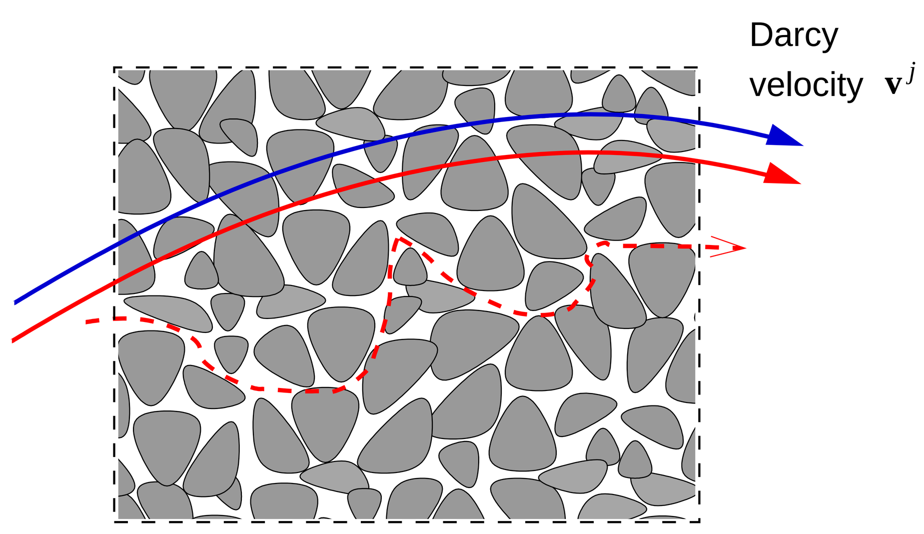


Figure 2: *Flow through a porous medium.*

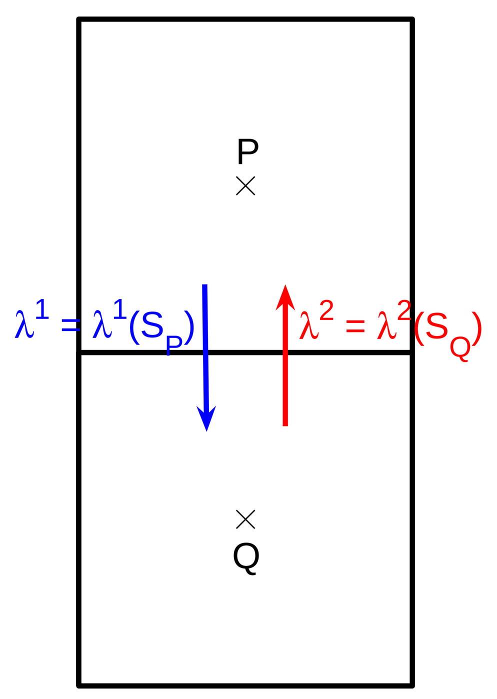


Figure 3: *Countercurrent flow.*

3. Potential ordering

The flow direction of phase j is completely determined by $\nabla\Phi^j$, where

$$\Phi^j = p - \rho^j \mathbf{g}^T\mathbf{x} \tag{3}$$

is the phase potential. If the control volumes are numbered in decreasing order of Φ^j, then the upwind discretization ensures

$$f_i^j \text{ is a function of } S_l^j \implies l \le i. \tag{4}$$

So the residual functions look like

$$\begin{aligned} f_1^j\,(S_1^j, \quad & p_1,\dots,p_N) = 0 \\ f_2^j\,(S_1^j, S_2^j, \quad & p_1,\dots,p_N) = 0 \\ & \vdots \\ f_N^j(S_1^j,\dots,S_N^j, & \, p_1,\dots,p_N) = 0 \end{aligned} \tag{5}$$

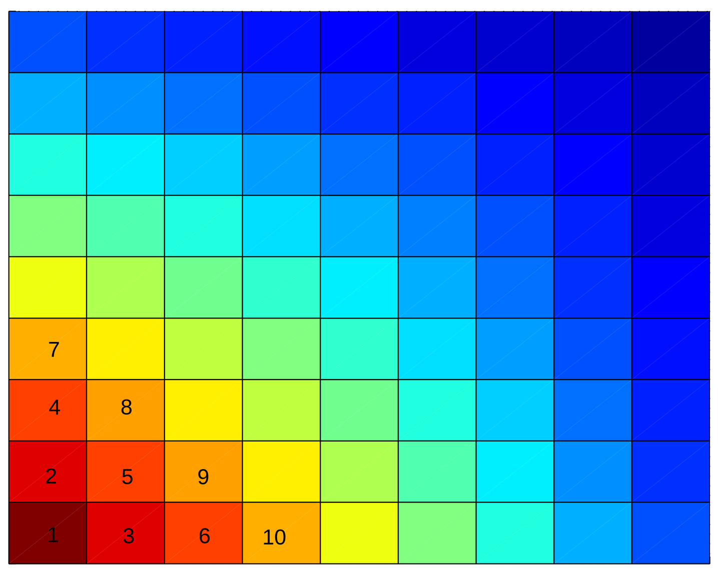


Figure 4: *Ordering from highest to lowest potential.*

4. Reduced-order Newton method

Suppose $[\partial\lambda^i/\partial S^j]$ is a lower-triangular matrix. Then the residual functions for all phases can be written as

$$\begin{array}{lll} f_1^1 & (S_1^1, & p_1,\dots,p_N) = 0 \\ f_2^1 & (S_1^1, S_2^1, & p_1,\dots,p_N) = 0 \\ \vdots & & \\ f_N^1 & (S_1^1,\dots,S_N^1, & p_1,\dots,p_N) = 0 \\ f_1^2 & (S_1^1,\dots,S_N^1, S_1^2, & p_1,\dots,p_N) = 0 \\ \vdots & & \\ f_N^{n-1} & (S_1^1,\dots,S_N^1,\dots,S_1^{n-1},\dots,S_N^{n-1}, & p_1,\dots,p_N) = 0 \\ \hline f_1^n & (S_1^1,\dots,S_N^1,\dots,S_1^{n-1},\dots,S_N^{n-1}, & p_1,\dots,p_N) = 0 \\ \vdots & & \\ f_N^n & (S_1^1,\dots,S_N^1,\dots,S_1^{n-1},\dots,S_N^{n-1}, & p_1,\dots,p_N) = 0. \end{array} \tag{6}$$

If $p_1,\dots,p_n$ are known, we can solve the first $(n-1)\times N$ equations for S_i^j by *forward substitution*. Thus, the first $(n-1)\times N$ equations can be seen as constraints that implicitly define the functions

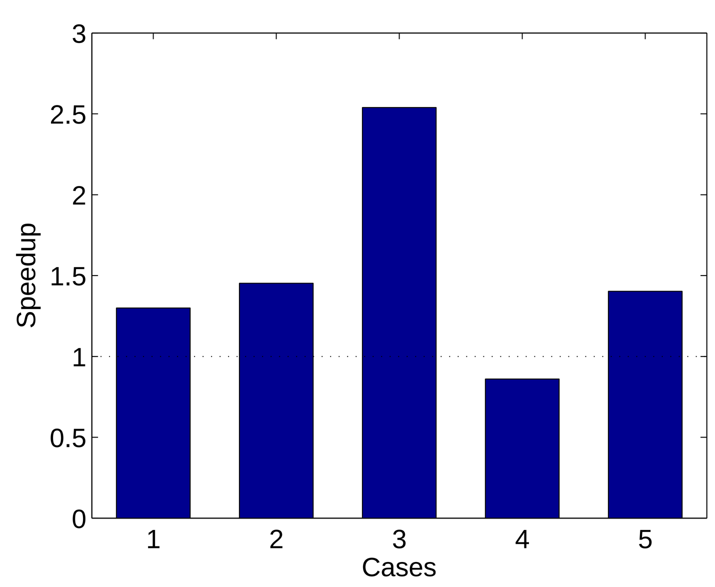
$$S_i^j = S_i^j(p_1,\dots,p_N). \tag{7}$$

The last N equations become

$$f_i^n(S(p_1,\dots,p_n), p_1,\dots,p_n) = 0, \qquad i = 1,\dots,N. \tag{8}$$

Newton's method is used to solve (8). The resulting method is more efficient because it avoids costly time-step cuts due to non-convergence. Global convergence can be proved for 2-phase incompressible 1D flow without gravity [3]:

1. If the λ^j are uniformly convex, then the reduced Newton method converges globally for **large** Δt;
2. Suppose the λ^j are convex. Then there exists a set of constraint equations (which can be chosen dynamically) such that the reduced Newton method converges **globally** for all Δt.



1. 2-phase, 3D flow (141,900 cells)
2. 2-phase, 3D flow (1,122,000 cells)
3. Same as #2 for large Δt
4. 3-phase, 2D flow (13,200 cells)
5. Same as #4 for large Δt

Figure 5: *Performance of reduced Newton method [2].*

5. Linear preconditioning

To solve the linear system within each Newton iteration, iterative methods (e.g. GMRES) are preferred for large-scale (esp. 3D) problems. The Constrained Pressure Residual (CPR) method [1] exploits the saturation-pressure coupling via the two-stage preconditioner

$$M^{-1} = M_2^{-1}(I - AM_1^{-1}) + M_1^{-1}, \tag{9}$$

which is derived from the stationary iteration

$$M_1 x^{(k+1/2)} = (M_1 - A)x^{(k)} + b, \tag{10}$$
$$M_2 x^{(k+1)} = (M_2 - A)x^{(k+1/2)} + b, \tag{11}$$

where

- M_1 = Elliptic solve on decoupled pressure system (AMG),
- M_2 = Block ILU preconditioner on global system.

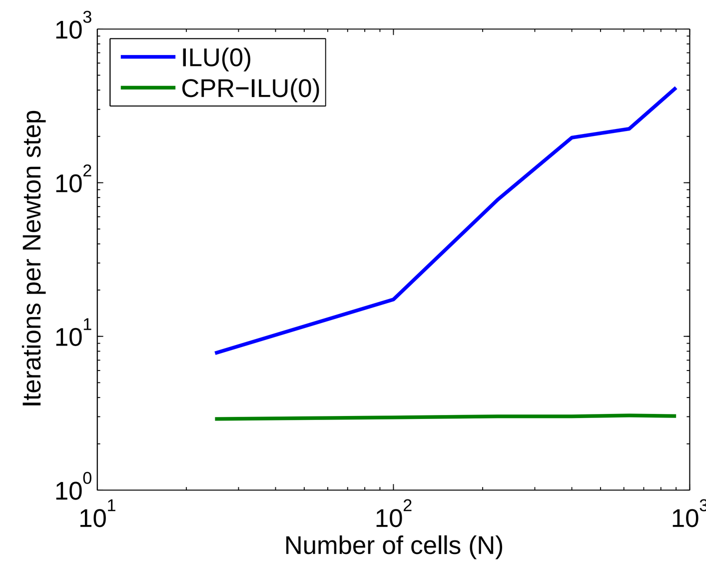
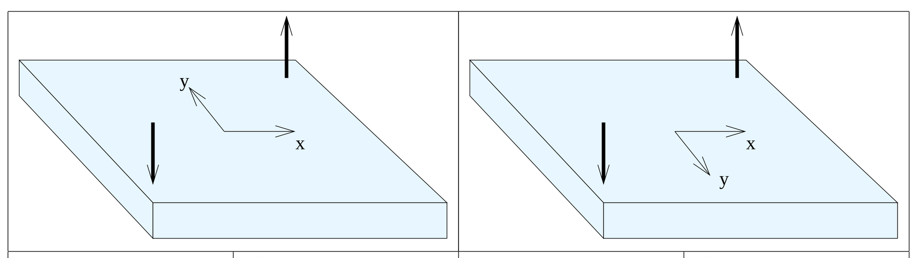


Figure 6: *CPR vs. ILU(0) for a simple 2-phase problem.*

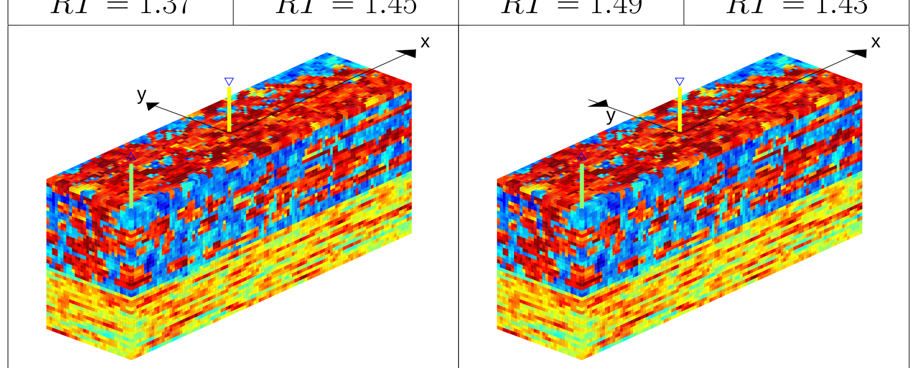
The above figure shows that CPR is nearly grid-independent, unlike ILU(0). For larger and more complex problems, CPR convergence can be improved by reordering the cells from **upstream to downstream** before computing ILU(0) factors for M_2. For cocurrent 2-phase flow [3]:

1. A block ILU(0) factorization exists whenever the cells are ordered from upstream to downstream, and it is exact with respect to saturation.
2. If L_1, U_1 and L_2, U_2 are two ILU factors computed based on two topological orderings of the same flow graph, then L_1, L_2 (also U_1, U_2) are identical up to permutation.

Hence the ordering is optimal and reduces the sensitivity of CPR with respect to flow configurations, as shown in the examples below.



Natural ord.	Potential ord.	Natural ord.	Potential ord.
$Its = 254$	$Its = 254$	$Its = 346$	$Its = 246$
$RT = 1.37$	$RT = 1.45$	$RT = 1.49$	$RT = 1.43$



Natural ord.	Potential ord.	Natural ord.	Potential ord.
$Its = 389$	$Its = 349$	$Its = 447$	$Its = 351$
$RT = 596$	$RT = 614$	$RT = 631$	$RT = 616$

Figure 7: *Ordering effects on CPR-ILU preconditioner. Its = Total GMRES iters., RT = Running time (sec).*

References

[1] J. R. Wallis, R. P. Kendall, and T. E. Little. *Constrained residual acceleration of conjugate residual methods*, In Eighth SPE Symposium on Reservoir Simulation, pp. 415–428, 1985.

[2] F. Kwok and H. Tchelepi. *Potential-based reduced Newton algorithm for nonlinear multiphase flow in porous media*, J. Comput. Phys 227, pp. 706–727, 2007.

[3] F. Kwok. *Scalable Linear and Nonlinear Algorithms for Multiphase Flow in Porous Media*, PhD thesis, Stanford University, Stanford, CA, Dec. 2007.

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