Plane wave discontinuous Galerkin methods

for homogeneous Helmholtz boundary value problems

Claude J. Gittelson

joint work with Ralf Hiptmair and Ilaria Perugia

Seminar for Applied Mathematics
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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

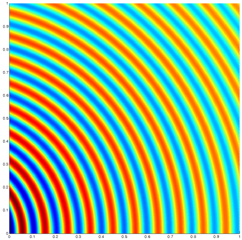
Seminar for Applied Mathematics

Helmholtz BVP and plane wave basis

Boundary value problem

For domain $\Omega \subset \mathbb{R}^2$,

$$\begin{aligned} -\Delta u - \omega^2 u &= 0 \qquad \text{in } \Omega \\ \nabla u \cdot \boldsymbol{n} + \mathrm{i}\omega u &= g \qquad \text{on } \partial\Omega \end{aligned}$$



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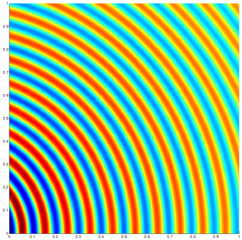

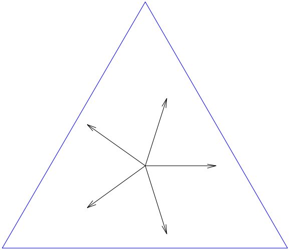

Plane wave basis

Mesh $\mathcal{T}_h$ of Ω.
Nonconforming test and trial space

$$V_h = \{\text{span of } p \text{ plane waves } e^{\mathrm{i}\omega \boldsymbol{d}\cdot\boldsymbol{x}} \text{ on each } K \in \mathcal{T}_h\}\ .$$

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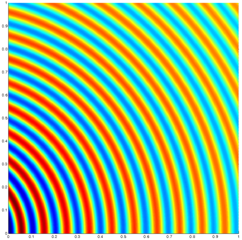

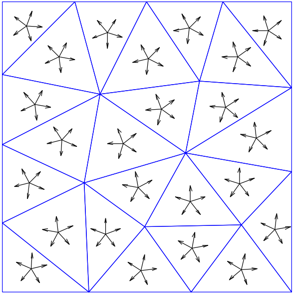

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PWDG method

Plane wave discontinuous Galerkin method

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\int_{\partial K} \widehat{u}_h \, \overline{\nabla v_h \cdot \boldsymbol{n}} \; - \mathrm{i}\omega \int_{\partial K} \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} \, \overline{v_h} \; = 0$$

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Trefftz method Basis functions in kernel of Helmholtz operator in every $K \in \mathcal{T}_h$

$\leadsto$ no volume integrals.

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Numerical fluxes

$$\widehat{u}_h(u_h) \approx u_h\big|_{\partial K} \qquad \widehat{\boldsymbol{\sigma}}_h(u_h) \approx \left.\frac{\nabla u_h}{i\omega}\right|_{\partial K}$$

weakly ensure interelement continuity and boundary conditions.

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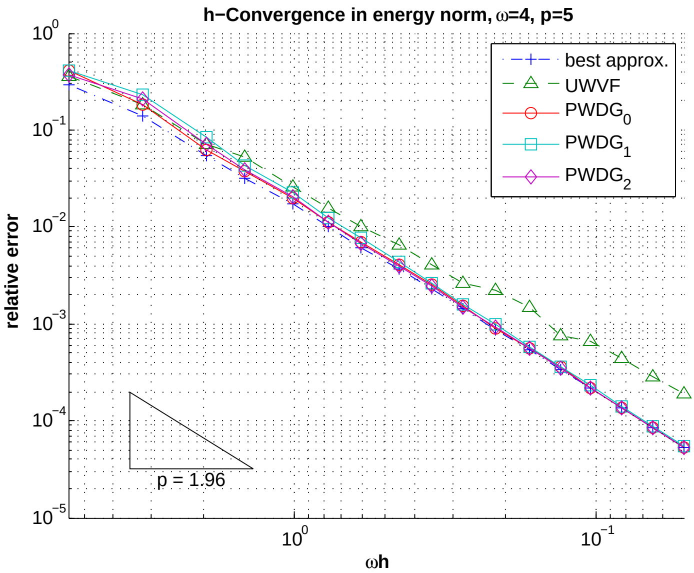
For a particular choice of $\widehat{u}_h$, $\widehat{\boldsymbol{\sigma}}_h$: recover ultra-weak variational formulation (UWVF) of O. Cessenat and B. Després.

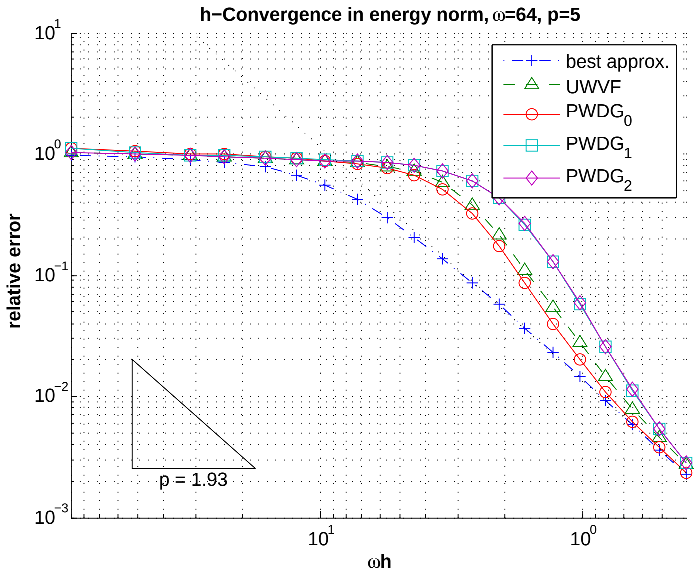
h-Convergence

Energy norm $$\|v\|_\omega^2 = \sum_{K\in\mathcal{T}_h} |v|_{H^1(K)}^2 + \omega^2 \|v\|_{L_2(\Omega)}^2$$

h-Convergence ($\omega = 4$)



h-Convergence ($\omega = 64$)



Pollution of discrete solution

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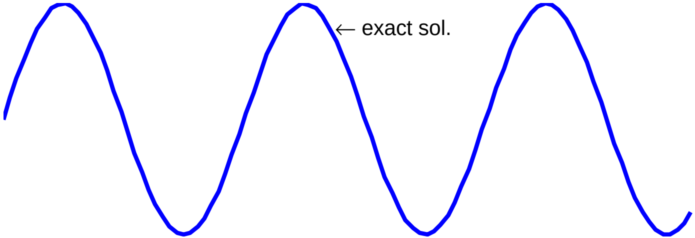

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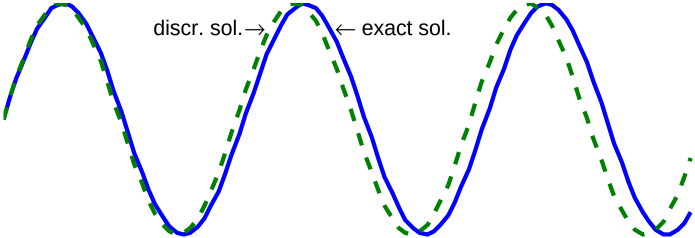

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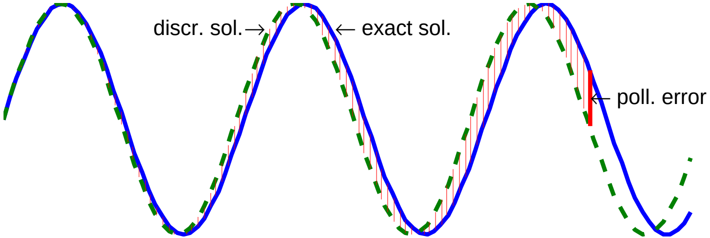

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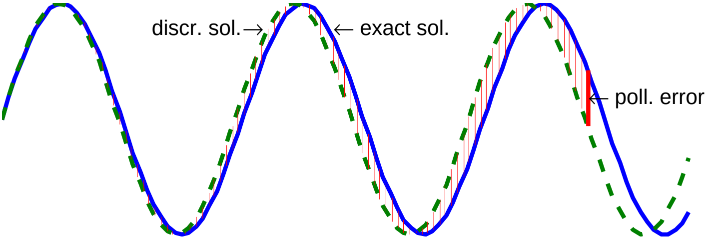

Error in energy norm (for odd $p \geq 3$, numerically determined):

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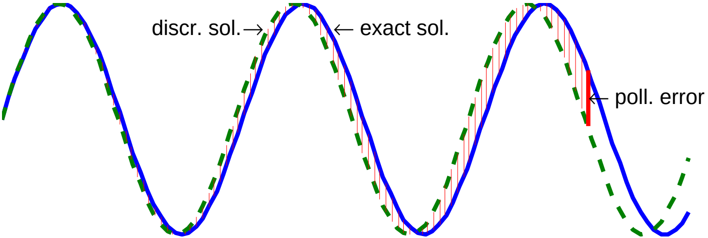

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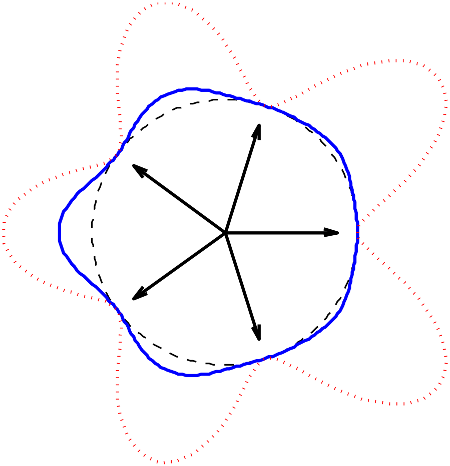
Dispersion $\mathrm{Re}(\widetilde{\omega} - \omega)$

Dissipation $\mathrm{Im}\,\widetilde{\omega}$



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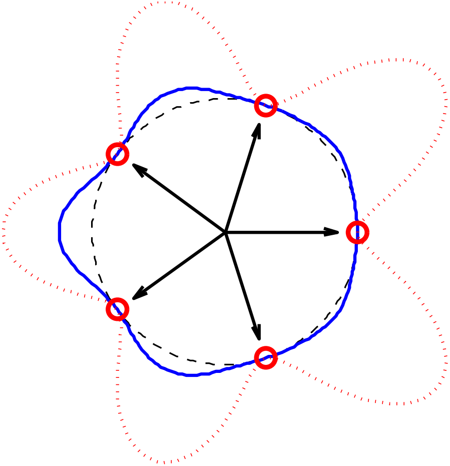

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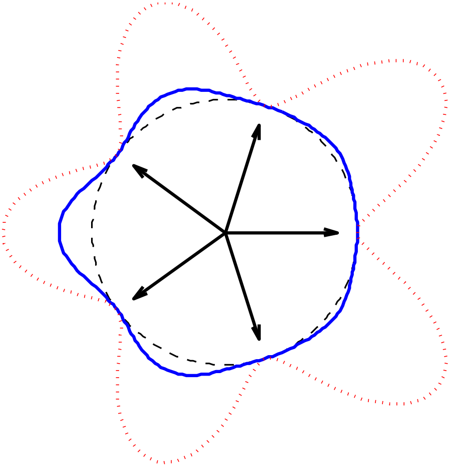

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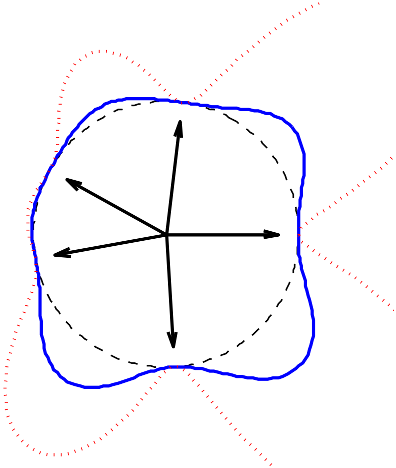
Adaptivity: Use dominant wave directions of exact solution!



Adaptive basis (possible approach)

Idea

Error $\quad e_h(\boldsymbol{x}) = u_h(\boldsymbol{x}) - u(\boldsymbol{x}) \approx c\,\mathrm{e}^{\mathrm{i}\omega \boldsymbol{d}\cdot\boldsymbol{x}} \quad$ for $\quad \boldsymbol{x} \in K$

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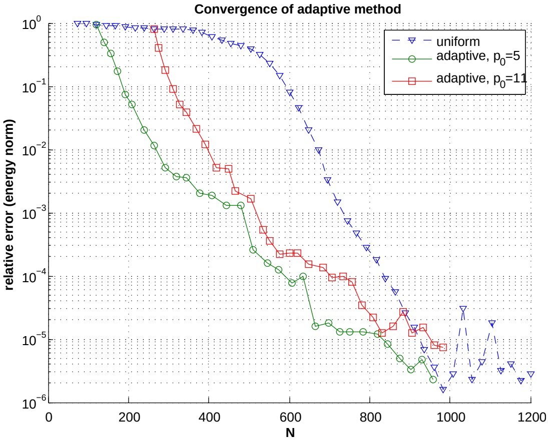
Propagation direction $\boldsymbol{d}_h$ of new basis function:

$$\widetilde{\boldsymbol{d}}_h = \mathrm{Re}\,\frac{1}{|K|}\int_K \frac{\nabla e_h}{\mathrm{i}\omega e_h}\,, \qquad \boldsymbol{d}_h = \frac{\widetilde{\boldsymbol{d}}_h}{|\widetilde{\boldsymbol{d}}_h|}$$

Convergence of adaptive method



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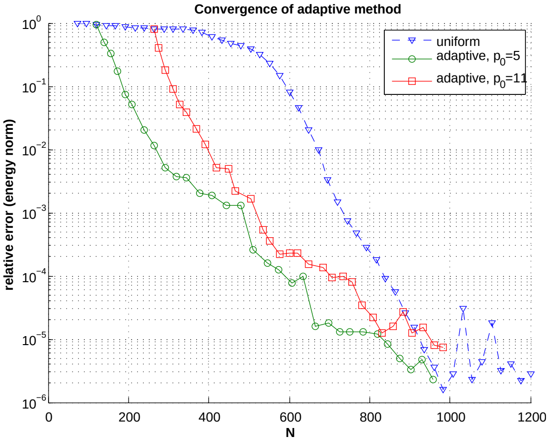

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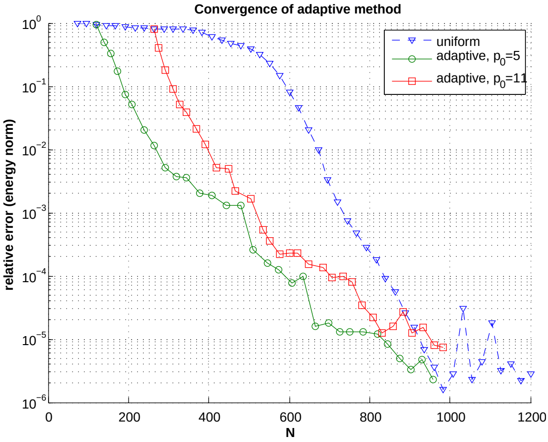

- Fast initial convergence
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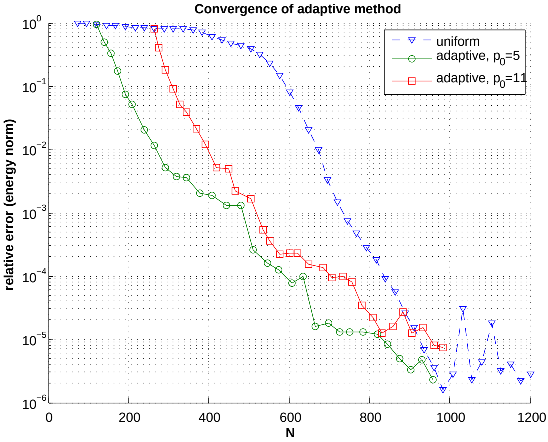

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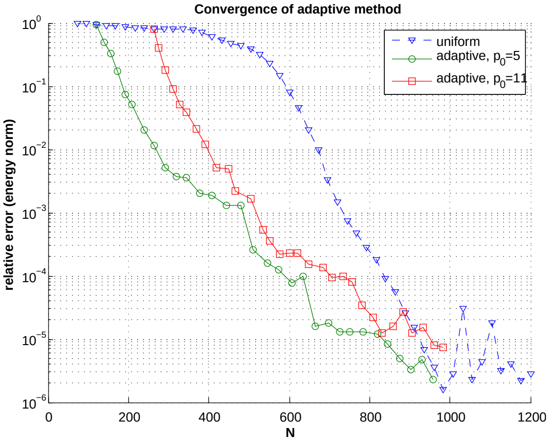

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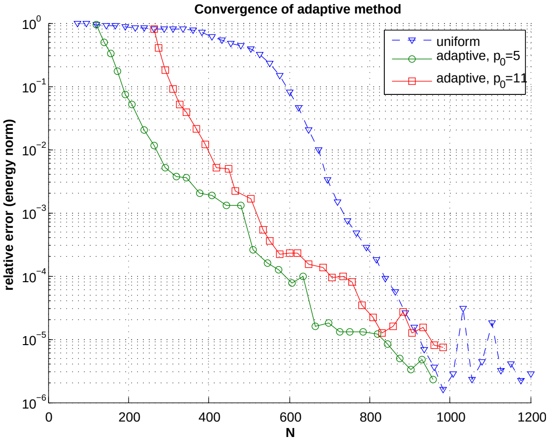

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However,

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